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Henri Poincaré Centennial Issue
Edited by Ferdinand Verhulst and Steven Wepster

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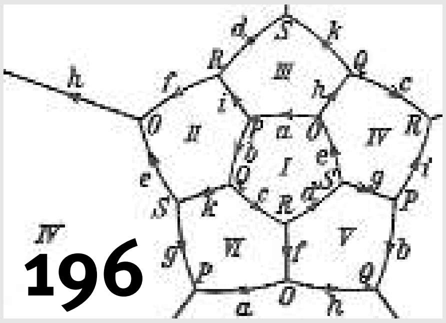
In the Spotlight

An interview with Henri Poincaré, made possible by permitting him to formulate the questions himself, on page 154.



Dirk Siersma describes the impact of Poincaré on algebraic topology, on page 196.

Colofon

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Koninklijk Wiskundig Genootschap

Het Koninklijk Wiskundig Genootschap (KWG) is een landelijke vereniging van beoefenaren van de wiskunde. In 1778 opgericht onder de zinspreuk: ‘Een onvermoeide arbeid komt alles te boven’ is het ’s werelds oudste nationale wiskundegenootschap. Leden van het KWG ontvangen het Nieuw Archief voor Wiskunde als onderdeel van hun lidmaatschap.
De contributie voor leden van het KWG bedraagt € 80 per jaar. Gepensioneerden en werklozen betalen € 40. Studenten ingeschreven aan een Nederlandse universiteit of hbo-opleiding, aio’s en oio’s kunnen lid worden voor € 30 per jaar. Studenten en pas afgestudeerden kunnen een jaar lang gratis lid worden van het KWG. Voor leden van de wiskundige vakverenigingen VVS, NVvW en NVORWO geldt het gereduceerd tarief van € 60.
Members of *SIAM*, the *Société Mathématique de France*, the *Gesellschaft für Angewandte Mathematik und Mechanik*, the *Deutsche Mathematiker-Vereinigung*, and of the *American, Australian, Belgian, Indian, London, Spanish,* and *South-African Mathematical Societies* living outside of the Netherlands pay € 60 instead of the full membership fee of € 80 a year. Conversely, these foreign societies, as well as the VVS and the NVORWO, have reduced membership fees for KWG-members. A postage charge of € 10 is added for members living abroad but in Europe, and a charge of € 12,50 for members living outside of Europe.
Instituten in Nederland en buitenland kunnen zich abonneren op het Nieuw Archief voor Wiskunde voor € 100 (Nederland), € 110 (Europa) of € 112,50 (buiten Europa).
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Informatie voor auteurs

Kopij
Het Nieuw Archief voor Wiskunde is een geredigeerd tijdschrift. Kopij dient elektronisch te worden aangeboden aan de hoofdredactie. Uitgebreide informatie en auteursinstructies kunt u vinden op www.nieuwarchief.nl.
Reprints
Auteurs ontvangen een elektronische reprint in pdf-formaat.
Volgende nummers

nummer	*verschijningsdatum*	*deadline kopij*
4	03–12–2012	verstreken
1	04–03–2013	01–11–2012
2	03–06–2013	01–02–2013
3	02–09–2013	01–05–2013

Instituutsleden

De publicaties van het Koninklijk Wiskundig Genootschap worden mede mogelijk gemaakt door de bijdragen van de volgende instituten.

Centrum Wiskunde & Informatica
Rijksuniversiteit Groningen
Universiteit van Amsterdam
Universiteit Leiden
Vrije Universiteit
Universiteit Utrecht
Technische Universiteit Eindhoven
Eurandom
Technische Universiteit Delft
Universiteit Twente
Radboud Universiteit Nijmegen
Freudenthal Instituut

On the cover

This special issue is devoted to Henri Poincaré (1854–1912), the famous French mathematician, who died one hundred years ago, on 17 July 1912.
Photo: Ernest Lebon, *Henri Poincaré: Biographie, bibliographie analytique des écrits*, Gauthier-Villars, Parijs, 1912 (2e ed.).

A special issue for Henri Poincaré

One hundred years ago, on 17 July 1912, Henri Poincaré died at the age of only 58 years. He had been unusually productive with fundamental results in mathematics, physics and astronomy, and he can be regarded as one of the most important mathematical physicists of all times; his work had a major impact throughout the twentieth century. He reshaped several existing fields and created some new ones in pure and applied mathematics. Highlights include his thesis which excels in the qualitative, geometric approach to differential equations, his analysis of automorphic functions that put him in contact with Fuchs and Klein, his creation of the theory of dynamical systems and his major influence on topology (analysis situs).

Poincaré also expressed philosophical ideas, and his philosophical essays made him well-known to the general public. They put him at the heart of the intellectual discourse of those times in France and other European countries. Apart from scientific papers and books, there exist thousands of letter exchanges between Poincaré and other scientists, family and friends. He attended dinner sessions organised by the psychoanalyst Marie Bonaparte where writers, composers and politicians met; Poincaré inspired Friedrich Nietzsche with his recurrence theorem. As chairman of the committee that had to assess the scientific merit of the 'proofs' for the guilt of the unfortunate Alfred Dreyfus, he took a view that was opposed to the ideas of the French reactionary establishment of those times, discarding the so-called evidence.

In this special NAW issue dedicated to Henri Poincaré we have been able to procure an exclusive interview with the eminent scientist, but apart from this, the emphasis is on his scientific work. It is not so well-known that Poincaré made fundamental contributions to the theory of partial differential equations, introducing for instance L^2-convergence (as it is known nowadays) and the idea of a generalised solution. We are happy that Jean Mawhin has written for us about this topic. We all know that Poincaré developed his general theory of dynamical systems with an eye on applications in celestial mechanics, but later he applied this theory to non-linear oscillations with applications in telegraph communication. Self-excited oscillations as studied later by Van der Pol can be found in the papers on this topic; Jean-Marc Ginoux discusses these results. Another aspect is the emergence of algebraic ideas around 1900; Jeremy Gray writes about Poincaré's use of group theory in mathematical physics.

In the nineteenth century the necessity of convergence of series had become a kind of mathematical dogma, so it looked slightly heretic when both Stieltjes and Poincaré introduced around 1885 the notion of approximation by asymptotic- or semi-convergent series; one of the articles describes this development. There was a relation with another famous Dutchman, L.E.J. Brouwer, who was inspired by Poincaré but carried the ideas of intuitionism much further; see the article by Van Dalen. The style of writing of Poincaré was one of permanent discourse with the reader and one cannot think of a style more opposed to this than the writing style of M. Bourbaki. Poincaré's most often used sentence is "ce n'est pas tout" after which he starts elucidating new aspects of the problem at hand. This is also typical for his creation of topology which is inspired by three-dimensional geometry and guided by intuition; see the article by Siersma.

Part of the mathematical ideas of Henri Poincaré are still unexplored, but there are also messages of a broader nature to be obtained from his work. First we note that geometry and analysis are intertwined in his papers. In dynamical systems for instance we note the interplay of geometric ideas, the qualitative aspects that give general insight, and the quantitative analysis resulting in actual formulas and predictions. This combination has been very fruitful. Secondly, we mention the way he presents mathematical and other scientific ideas to the specialist and to the general public. His style of writing produces an efficient transfer of ideas and it helps to communicate to the public the excitement, the interest and the elegance of scientific reasoning. We remind the readers of this journal of the perpetual importance of communicating the excitement of doing mathematics, in particular towards prospective students.

We cannot do justice to all of Poincaré's achievements in a single issue of this journal, but fortunately this is remedied by three biographies that appear in English this year. These are the first modern biographies. The books are so recent that there was no time to have them properly reviewed for this special issue, therefore we asked their authors to highlight the key ideas and contents of each. There is unavoidably some overlap between these biographies but as a whole they supplement each other nicely. With a scientist of the stature of Henri Poincaré, there is room for even more biographies.

Ferdinand Verhulst and Steven Wepster (eds.)
Mathematical Institute, Utrecht University

Servicepagina

Service Page

Koninklijk Wiskundig Genootschap

❑ **Secretariaat**
Joke Blom, CWI
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❑ **Wintersymposium**
Het KWG organiseert jaarlijks, op de eerste zaterdag in januari, een symposium voor docenten wiskunde in het voortgezet onderwijs en andere belangstellenden. Doel van het symposium is contacten tussen wiskundedocenten en beroepsmatige beoefenaars van wiskunde te stimuleren, om zo leerlingen een beter beeld te geven van de rijkdom aan mogelijkheden van wiskunde. Thema's van de afgelopen jaren waren: 'Grootschalig rekenen en gezondheidszorg', 'Logica', 'Blik op oneindig', 'Wiskunde een Kunst', 'Wiskundehelden uit de Gouden Eeuw'. In 2013 is het thema 'Wiskunde met Natuurkunde'. De sprekers zijn Henk Broer (RUG), Ute Ebert (CWI) en Erik Verlinde (UvA). www.wiskgenoot.nl/watbiedt/

❑ **Stieltjes Prijs voor Jop Briët**
Vanaf 1996 wordt jaarlijks een Stieltjes Prijs toegekend (tot 2009 door het toenmalige Thomas Stieltjes Institute for Mathematics, vanaf 2010 door WONDER) voor het beste proefschrift in de wiskunde, dat in dat betreffende jaar is verdedigd aan een Nederlandse universiteit. De prijs bedraagt 2.500 euro, beschikbaar gesteld door de Stichting Compositio Mathematica. De Stieltjesprijs 2011 is op 18 juli toegekend aan Jop Briët. Jop heeft zijn proefschrift *Grothendieck Inequalities, Nonlocal Games and Optimization*, verdedigd bij de Universiteit van Amsterdam op 27 oktober 2011; promotor was prof.dr. H.M. Buhrman.

Platform Wiskunde Nederland

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Science Park 123, L013, 1098 XG Amsterdam
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Tel: 020-5924006/06-51892525
www.platformwiskunde.nl

❑ **PWN Nieuwsbrief**
PWN publiceert maandelijks een PWN Nieuwsbrief, die op de website te vinden is. Actuele PWN informatie wordt verder zowel via de website als via Twitter gegeven.

European Mathematical Society

❑ **Council Meeting van de EMS**
Het hele verslag van de EMS Council Meeting van 30 juni en 1 juli 2012 in Krakow is te vinden op www.euro-math-soc.eu/node/2744.

De Council heeft besloten dat 7ECM van 2016 plaats zal vinden in Berlijn. De lidmaatschapsbijdrage, ongewijzigd sinds 2008, zal met 4–5% omhoog gaan. Voor de periode 2013–2016 werden Franco Brezzi en Martin Raussen verkozen tot vice-president en Laurence Halpern, Gert-Martin Greuel, Alice Fialowski en Armen Sergeev tot leden van de Executive Committee. De Council werd afgesloten met een paneldiscussie over procedures voor het publiceren en over een 'Code of Practice', ontworpen door de Ethics Committee van de EMS.

❑ **Medewerkers EMS-website gezocht**
De EMS zoekt voor haar website hulp bij het opzetten van een tweede team van boekrecensenten en een team van blogmoderators, en bij het verder ontwikkelen van de website. www.euro-math-soc.eu

Recente publicaties van het KWG

❑ **Indagationes Mathematicae** (www.elsevier.com/locate/indag)
Special issue devoted to: Floris Takens (1940–2010); S. van Strien and H. Broer (eds.), Volume 22, issues 3–4 (gratis toegankelijk op www.sciencedirect.com).

❑ **Epsilon Uitgaven** (www.epsilon-uitgaven.nl)

72. Handboek wiskundedidactiek, Paul Drijvers, A. van Streun, B. Zwaneveld, €34,00, 2012.
71. Kansrekening – een introductie, A.J. van den Brandhof, €17,00, 2012.
70. Wiskunde in Werking van A naar B, M. de Gee, €34,00, 2011.
69. De Riemann-hypothese – een miljoenenprobleem, R. van der Veen en J. van de Craats, €19,00, 2011.
66. Intuïtionistische Analyse – een constructief denkraam, D. van Dalen, €21,00, 2011.

❑ **Zebra-reeks** (Epsilon Uitgaven)

34. De Ster van de dag gaat op en onder – rekenen aan zonsopkomst en zonsondergang, A. Goddijn, €10,00, 2012.
33. Ontwikkelen met Kettingbreuken, M. Kindt en P.W.H. Lemmens, €10,00, 2011.

Manuscripten voor Epsilon Uitgaven kunt u mailen naar c.boss@epsilon-uitgaven.nl.

Agenda | Upcoming Events

september 2012

14 september 2012
❑ **Finale Nederlandse Wiskunde Olympiade**
Finale van de jaarlijkse scholierenwedstrijd, waarbij het Nederlandse team voor de Internationale Wiskunde Olympiade 2013 zal worden geselecteerd
plaats *TU/e, Eindhoven*
info *www.wiskundeolympiade.nl*

17 – 21 september 2012
❑ **Active Dynamics on Microscales: Molecular Motors and Self-Propelling Particles**
Workshop over dynamica op microscopische schaal, met name zichzelf voortbewegende eiwitten
plaats *Lorentz Center@Oort, Leiden*
info *www.lorentzcenter.nl*

20 september 2012
❑ **Henri Poincaré Centennial Symposium**
Symposium met diverse binnen- en buitenlandse sprekers ter gelegenheid van de 100ste sterfdag van Henri Poincaré
plaats *De Uithof, Utrecht*
info *www.staff.science.uu.nl/˜wepst101/poincare.html*

20 september 2012
❑ **Eerste Nederlandse Ig Nobel Night**
Nacht voor onderzoekers en wetenschappers met Nederlandse Ig Nobelwinnaars, en live vertoning van de Ig Nobel Prize ceremonie
plaats *Stadsgehoorzaal, Leiden*
info *www.ignobelnight.nl*

21 september 2012
❑ **Wiskundetournooi**
Jaarlijkse wiskundewedstrijd voor teams van vijf middelbare scholieren uit de bovenbouw
plaats *Radboud Universiteit, Nijmegen*
info *www.ru.nl/wiskundetoernooi*

27 september 2012
❑ **Bèta onder de Dom 2012**
Workshops over de nieuwe curricula voor natuurkunde, scheikunde en biologie, en de nieuwe NLT-modules
plaats *De Uithof, Utrecht*
info *www.betasteunpunt-utrecht.nl/?pid=11*

28 september 2012
❑ **Discovery Festival**
Uitgaansnacht-met-inhoud vol wetenschap, muziek, kunst en experimenten
plaats *NEMO, Amsterdam*
info *www.discoveryfestival.nl*

oktober 2012

3 – 5 oktober 2012
❑ **Woudschoten Conferentie**
Jaarlijkse conferentie van de Nederlandse en Vlaamse numerieke wiskundige gemeenschappen
plaats *Woudschoten Conferentiecentrum, Zeist*
info *wsc.project.cwi.nl/conferentieN.php*

4 oktober 2012
❑ **Opening LorentzCenter@Snellius**
Officiële opening van de nieuwe locatie van het Lorentz Center
plaats *LorentzCenter@Snellius, Leiden*
info *www.lorentzcenter.nl*

6 oktober 2012
❑ **Open dag Science Park Amsterdam**
Diverse instituten gevestigd in het Science Park Amsterdam openen hun deuren
plaats *Science Park, Amsterdam*
info *www.scienceparkamsterdam.nl*

6 oktober 2012
❑ **Night of the Nerds**
Avond met workshops, clinic en experimenten voor jongeren van 16 tot 20 jaar
plaats *NEMO, Amsterdam*
info *www.nightofthenerds.nl*

11 oktober 2012
❑ **Afscheidssymposium Jan van Mill**
Symposium voor algemeen wiskundig publiek met onder meer Ieke Moerdijk, Henk Broer, Geurt Jongbloed, Jan Dijkstra, Wistar Comfort en Alan Dow
plaats *VU, Amsterdam*
info *www.math.vu.nl/˜aad/vanmill.html*

Suggesties voor deze agenda zijn welkom.
Redacteur: Martijn Zaal
agenda@nieuwarchief.nl

12 oktober 2012

❑ **Junior Wiskunde Olympiade 2012**

Jaarlijkse wiskundewedstrijd voor scholieren uit de onderbouw

plaats *VU, Amsterdam*

info *www.wiskundeolympiade.nl/junior*

15 – 19 oktober 2012

❑ **The Future of Phylogenetic Networks**

Workshop over het bestuderen van netwerkmodellen die in de evolutietheorie worden gebruikt

plaats *Lorentz Center@Oort, Leiden*

info *www.lorentzcenter.nl*

20 oktober 2012

❑ **Nerds on Stage**

Interactief programma voor jongeren van 16 tot 24 jaar over wetenschap en techniek

plaats *Rotterdamse Schouwburg, Rotterdam*

info *www.nerdsonstage.nl*

24 – 26 oktober 2012

❑ **Openingssymposium Eindhoven Multiscale Institute**

Officieel openingssymposium van het Eindhoven Multiscale Institute met diverse internationale sprekers

plaats *TU/e Eindhoven*

info *www.tue.nl/onderzoek*

26 oktober 2012

❑ **Springer day**

Dag ter nagedachtenis van T.A. Springer, met Michel Brion (Grenoble), Arjeh Cohen (Eindhoven), Corrado de Concini (Rome) en Eric Opdam (UvA)

plaats *Universiteit Utrecht*

info *www.staff.science.uu.nl/~looij101/Springer_day.html*

november 2012

1 – 3 november 2012

❑ **YEQT-VI**

Jaarlijkse workshop voor jonge onderzoekers in wachtrijtheorie, met dit jaar als thema 'analytic methods in queueing systems'

plaats *Eurandom, Eindhoven*

info *www.eurandom.nl/events*

12 – 14 november 2012

❑ **41ste bijeenkomst Stochastici**

Jaarlijkse bijeenkomst van Nederlandse stochastici en statistici

plaats *Congrescentrum de Werelt, Lunteren*

info *www.cwi.nl*

12 – 16 november 2012

❑ **Multiscale Systems Biology of Cancer**

Workshop over de bestudering van tumoren op orgaan-, cel- en moleculair niveau

plaats *Lorentz Center@Oort, Leiden*

info *www.lorentzcenter.nl*

14 – 18 november 2012

❑ **Upscaling Techniques and Homogenization**

Korte cursus over schalings- en homogenisatietechnieken voor modellen in een complex domein of met sterk oscillerende eigenschappen

plaats *TU/e, Eindhoven*

info *www.win.tue.nl/~pop/JMBCCourse.html*

18 november 2012

❑ **Waarom drijf je in de Dode Zee?**

Kinderlezing door natuurkundige Marcel Vreeswijk over dichtheid, drijven en zinken

plaats *NEMO, Amsterdam*

info *www.kinderlezingen.nl*

29 november 2012

❑ **DIAMANT symposium**

Halfjaarlijks symposium van het DIAMANT onderzoekscluster

plaats *Kaap Doorn Conferentiecentrum, Doorn*

info *websites.math.leidenuniv.nl/diamant*

december 2012

14 – 18 december 2012

❑ **Representing stream**

Workshop over de bestudering van datastromen door wiskundigen en informatici

plaats *Lorentz Center@Snellius*

info *www.lorentzcenter.nl*

20 december 2012

❑ **Studiedag voor wiskundeleraren**

Studiedag met lezing en workshops voor wiskundeleraren met als thema 'Wiskunde? Opheffen! (de vaardigheid)'

plaats *Bernoulliborg, Grongingen*

info *www.rug.nl/wiskunde*

januari 2013

14 – 18 januari 2013

❑ **Random Polymers**

Workshop over random polymeren voor onderzoekers in kansrekening, combinatoriek, natuurkunde en scheikunde

plaats *Eurandom, Eindhoven*

info *www.eurandom.nl/events*

21 – 23 januari 2013

❑ **12th Winter school on Mathematical Finance**

Jaarlijkse winterschool Mathematical Finance met lezingen en mini-cursussen

plaats *Congrescentrum De Werelt, Lunteren*

info *staff.science.uva.nl/~spreij/winterschool/winterschool.html*

21 – 31 januari 2013

❑ **Eerste ronde Nederlandse Wiskunde Olympiade**

De eerste ronde van de Nederlandse Wiskunde Olympiade 2013, dit jaar voor het eerst met een flexibele datum

plaats *Deelnemende scholen*

info *www.wiskundeolympiade.nl*

28 januari – 1 februari 2013

❑ **SWI 2013**

Jaarlijkse studiegroep waarbij wiskundigen werken aan problemen uit de industrie

plaats *Lorentz Center, Leiden*

info *websites.math.leidenuniv.nl/SWI-2013/home.html*

31 januari – 1 februari 2013

❑ **Panamaconferentie**

Jaarlijkse conferentie over de ontwikkeling van reken- en wiskundeonderwijs

plaats *De Uithof, Utrecht*

info *www.fisme.science.uu.nl/panama*

Nieuws

| News

P=NP toch niet bewezen

Onderzoekers van het CWI in Amsterdam, de Vrije Universiteit Brussel en de Friedrich-Alexander Universität Erlangen-Nürnberg zijn erin geslaag een 26 jaar oude bewijspoging over P=NP definitief te weerleggen.

Een van de grootste onopgeloste problemen in de wiskunde is de vraag hoe lastig het zogenaamde handelsreizigersprobleem is. Het handelsreizigersprobleem draait om de vraag in welke volgorde een reiziger gegeven locaties kan bezoeken om zo min mogelijk afstand af te hoeven leggen. In veel hedendaagse logistieke vraagstukken speelt het handelsreizigersprobleem, of een variatie erop, een belangrijke rol. Daarnaast is het probleem ook van theoretisch belang: het probleem behoort tot de klasse van NP-complete problemen. Als er voor dit probleem een oplosmethode bestaat dan is meteen bewezen dat alle problemen relatief snel op te lossen zijn met de computer, een stelling die ook wel wordt samengevat als 'P=NP'.

In 1986 claimde de Canadees Ted Swart dat het handelsreizigersprobleem snel kon worden opgelost met een lineair programma. Al na een jaar werd deze claim weerlegd door Mihalis Yannakakis: lineaire programma's zoals die van Swart waren niet in staat het handelsreizigersprobleem snel op te lossen. Nu, 26 jaar later, is ook bewezen dat zogenaamde niet-symmetrische lineaire programma's het handelsreizigersprobleem niet snel op kunnen lossen. *www.cwi.nl*

Jan Hogendijk wint eerste Otto Neugebauer Prijs

De eerste Otto Neugebauer Prijs is toegekend aan Jan Hogendijk van de Universiteit Utrecht. De Otto Neugebauer Prijs wordt toegekend aan vernieuwend en invloedrijk werk over de geschiedenis van de wiskunde. De prijs bestaat uit een certificaat en een geldbedrag van €5000, beschikbaar gesteld door uitgeverij Springer.

Jan Hogendijk kreeg de prijs 'for having illuminated how Greek mathematics was absorbed in the medieval Arabic world, how mathematics developed in medieval Islam, and how it was eventually transmitted to Europe'. De prijs is uitgereikt tijdens het zesde European Congress of Mathematics van 2 tot 6 juli in Krakau. *www.6ecm.pl/en/ems-prizes*

Khan Academie biedt uitkomst voor onderwijs

Door de vele bezuinigingen gaat het onderwijs in Nederland achteruit. Er zijn minder leraren en er zijn plannen voor grotere klassen. Dit gaat ten koste van de ontwikkeling van de leerlingen. De Khan Academie wil hiervoor een oplossing bieden door gratis onderwijs aan te bieden. Daarnaast biedt de Khan Academie ook een methode voor leraren om een beeld te krijgen van de struikelpunten van zijn leerlingen, om vervolgens gericht les te kunnen geven.

De Khan Academie is een online onderwijsplatform gebaseerd op de Amerikaanse Khan Academy (www.khanacademy.org), die is opgezet door Salman Khan, een Amerikaanse wiskundeleraar. Via Skype gaf hij wiskundebijles aan zijn nichtje, die aan de andere kant van Amerika woonde. Doordat ze niet altijd tegelijk online konden zijn nam hij de bijlessen op en plaatste ze op YouTube. Binnen de kortste keren werd hij ook bij anderen razend populair, wat leidde tot de oprichting van de Kahn Academy, zodat iedereen online gebruik kan maken van gratis onderwijs.

De Nederlandse Khan Academie is eind 2011 begonnen. De organisatie houdt zich momenteel vooral bezig met het vertalen van bestaande Engelstalige oefeningen en uitlegvideo's naar het Nederlands. De

Deze rubriek is een kroniek van wiskundige activiteiten in Nederland. Toekomstige activiteiten worden aangekondigd en van voorbije activiteiten wordt verslag gedaan. Wilt u uw aankondiging of verslag in deze rubriek geplaatst zien? Stuur ons dan uw bijdrage van ± 350 woorden, zo mogelijk met illustratie. De redactie behoudt zich het recht voor berichten te weigeren of in te korten.

Redacteur: Martijn Zaal

nieuws@nieuwarchief.nl

focus ligt op het aanbieden van online reken- en wiskundelessen. In de toekomst wil de Khan Academie dit gaan uitbreiden met andere vakken, waaronder taal.

Naast de opbouw van de website gaat de Khan Academie volgend schooljaar een aantal pilots doen met scholen van verschillende niveaus. Leraren maken zelf uitlegvideo's van de lesstof, waarmee de leerlingen vervolgens aan de slag kunnen. De leraar kan via de website nauwkeurig bijhouden welke leerling welke opdrachten heeft gemaakt en hoe de leerling heeft gescoord. Op deze manier weet de leraar welke leerlingen op welke vlakken extra aandacht nodig hebben en kan hij zijn lessen effectiever indelen.

Via www.khanacademie.nl/helpmee kan iedereen een bijdrage leveren aan de ontwikkeling van de Khan Academie. *www.khanacademie.nl*

Vijf Vidi's voor wiskunde

NWO heeft aan 94 vernieuwende wetenschappers een Vidi-beurs van € 800.000 toegekend. Dit jaar was het honoreringspercentage lager dan ooit: er waren maar liefst 652 voorstellen ingediend.

Onder de gehonoreerde aanvragen waren vijf voorstellen over wiskunde. Nikhil Bansal (TU Eindhoven) werkt aan wiskundige technieken om beslissingen te nemen die de negatieve effecten van onverwachte gebeurtenissen in de toekomst beperken. Gil Cavalcanti (Universiteit Utrecht) doet onderzoek naar vierdimensionale ruimten en gegeneraliseerde complexe structuren, twee gebieden uit de meetkunde die van belang zijn voor de natuurkunde. Het voorstel van Daan Crommelin (CWI) gaat over de relatie tussen kleine stromingpatronen in de oceaan die samen belangrijke invloed hebben op de grootschalige stromingen en daarmee op het klimaat. Het onderzoek van Michiel Hochstenbach (TU Eindhoven) gaat over de ontwikkeling van snellere en betrouwbaardere methoden om matrixproblemen op te lossen. Ronald van Luijk (Universiteit Leiden) kreeg een beurs voor onderzoek naar geheeltallige oplossingen van bepaalde klassen vergelijkingen. *www.nwo.nl*

Recordopbrengst voor boekenveiling

De internetboekenveiling van het Wereldwiskunde Fonds die eindigde op 31 mei heeft € 919,50 opgebracht. De opbrengst van de veiling komt ten goede aan het Wereldwiskunde Fonds dat wiskundeonderwijs in ontwikkelingslanden ondersteunt.

De volgende veiling zal worden gehouden in oktober. Ondertussen kunnen er boeken worden aangeboden of gekocht via www.wereldwiskundeboeken.nl. *www.wereldwiskundeboeken.nl*

Robbert Dijkgraaf geridderd tijdens Akademiemiddag KNAW

Tijdens de jaarlijkse akademiemiddag op 5 juni maakte premier Mark Rutte bekend dat Robbert Dijkgraaf benoemd is tot Ridder in de Orde van de Nederlandse Leeuw. De premier prees niet alleen de wetenschappelijke prestaties van Dijkgraaf maar ook zijn rol als brug tussen wetenschap en samenleving. Ook zei Rutte dat Nederland trots kan zijn op haar wetenschap, iets wat wordt ondersteund door de nieuwe functie van Dijkgraaf: sinds 1 juli is hij president van het Institute for Advanced Study in Princeton, een van de meest prestigieuze wetenschappelijke posities.

In zijn jaarrede benadrukte Dijkgraaf de noodzaak van vrijheid in de wetenschap, en het belang van 'nutteloos' onderzoek. Hij haalde de relatief sterke economische positie van bijvoorbeeld Zweden, Denemarken en Duitsland, landen die een 'inspirerend kennisbeleid' voeren, aan als bewijs dat fundamenteel onderzoek bijdraagt aan welzijn en welvaart.

Hans Clevers, die tijdens de akademiemiddag als nieuwe president van de KNAW werd geïnstalleerd, bracht hetzelfde punt onder de aandacht. Hij gaf aan zich in Den Haag sterk te gaan maken voor de positie van wetenschap en zich in te zetten voor een steviger rol van wetenschap in de landelijke politiek. *www.knaw.nl*

Nederlands succes op Internationale Wiskunde Olympiade

Nederland heeft geschiedenis geschreven bij de Internationale Wiskunde Olympiade in Argentinië door twee gouden medailles binnen te halen. Van de 42 keer dat Nederland heeft meegedaan, heeft het slechts twee keer eerder een gouden medaille gewonnen: in 1983 en in 1977. Het team, bestaande uit de zes beste middelbare scholieren van Nederland, won in Argentinië verder drie bronzen medailles en een eervolle vermelding. Nederland werd daarmee 22ste van de honderd deelnemende landen en liet alle andere West-Europese landen achter zich.

De Internationale Wiskunde Olympiade vond van 9 tot 15 juli plaats in Mar del Plata, Argentinië. Op elk van de twee wedstrijddagen kregen de leerlingen drie opgaven van hoog wiskundig niveau voor hun kiezen, waar ze vierenhalf uur aan konden werken. Voor elke opgave waren zeven punten te behalen. Jeroen Huijben (16) en Jetze Zoethout (17) wisten beiden maar liefst vier van de zes opgaven foutloos op te lossen, een uitzonderlijke prestatie. Ze behaalden daarmee elk een gouden medaille. Daarnaast behaalden Jeroen Winkel (15), Matthijs Lip (16) en Guus Berkelmans (18) een bronzen medaille. Michelle Sweering (15) behaalde een eervolle vermelding.

In totaal ontvingen van de 548 deelnemers 138 een bronzen medaille, 88 een zilveren medaille en 51 een gouden medaille. Het officieuze landenklassement werd aangevoerd door Zuid-Korea, China en de Verenigde Staten. *wiskundeolympiade.nl*

Foto: Birgit van Dalen

V.l.n.r. Michelle Sweering, Guus Berkelmans, Jetze Zoethout, Jeroen Huijben, Jeroen Winkel en Matthijs Lip op het strand in Mar del Plata met hun medailles

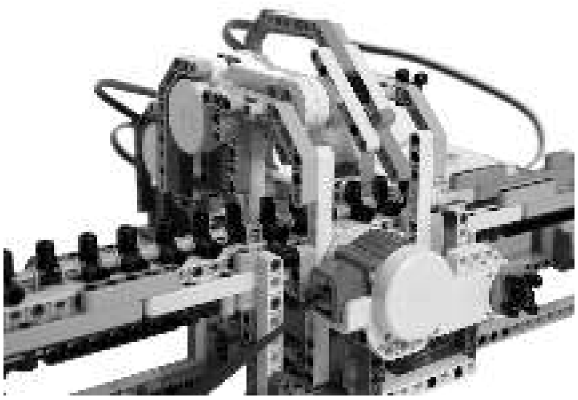
Illusionistisch werk van schilder Jos de Mey

In Slot Clemenswerth in Sögel (Duitsland) worden onder de titel 'Illusionistische Schilderkunst' vijftig kunstwerken van schilder Jos de Mey tentoongesteld. Het werk van Jos de Mey gaat vooral over zogenaamde ‘onmogelijke objecten’, objecten die alleen kunnen worden afgebeeld maar niet werkelijk kunnen bestaan. In zijn werk zijn invloeden te herkennen van onder andere Escher, Magritte en Brueghel: details van deze schilders zijn verwerkt in de omgeving van de onmogelijke constructies. De tentoonstelling is nog tot 31 oktober te bezoeken. *www.optische-fenomenen.nl*

Tentoonstelling ‘Turings Erfenis’

23 juni 2012 was de honderdste geboortedag van de beroemde Britste wiskundige Alan Turing. Ter ere hiervan is 2012 uitgeroepen tot het Alan Turingjaar, en heeft het CWI in Amsterdam de tentoonstelling ‘Turings Erfenis’ georganiseerd.

Turing wordt algemeen erkend als de grondlegger van de informatica. Zijn bekendste prestatie is het breken van de Enigma-code in de Tweede Wereldoorlog. De tentoonstelling over het leven en werk van Alan Turing laat zien hoe zijn ideeën vandaag de dag nog worden toegepast in onderzoek. De werking van de computer wordt geïllustreerd door een Turingmachine van LEGO, gebouwd door twee onderzoekers van het CWI. Op de tentoonstelling zijn diverse historische rekenmachines en computers te bewonderen, waaronder een originele Enigma-machine. De tentoonstelling is nog te bezoeken tot 6 oktober 2012. *www.cwi.nl*



Turingmachine van LEGO

Zilveren Tulp voor wiskundemeisjes

Het boek *Ik was altijd heel slecht in wiskunde* van wiskundemeisjes Jeanine Daems en Ionica Smeets is bekroond met een Zilveren Tulp. De stichting collectieve propaganda van het Nederlandse boek kent jaarlijks maximaal tien Zilveren Tulpen toe aan informatieve boeken die in hun genre een rol als ambassadeur vervullen richting het algemene Nederlandse publiek.

De jury: “Het boek maakt wiskunde minder elitair en heeft een heel toegankelijke vormgeving. De jury vindt het boek een origineel concept door de vele voorbeelden uit het dagelijks leven en verschillende opdrachten.” *cpnb.nl*

Wiskundige ontwikkelt nieuwe dodehoekspiegel

Wiskundige R. Andrew Hicks (Drexel University, Philadelphia) heeft een nieuwe dodehoekspiegel ontwikkeld. Traditionele platte spiegels, in de Verenigde Staten verplicht, geven de bestuurder van een voertuig een goede indruk van de grootte en afstand van andere voertuigen in de spiegel, maar hebben een klein gezichtsveld. De meeste gebogen spiegels geven een groter gezichtsveld, maar vervormen ook het beeld, waardoor andere voertuigen verder weg lijken te zijn dan daadwerkelijk het geval is.

De nieuwe spiegel combineert de voordelen van de twee spiegels: het gezichtsveld is veel groter dan dat van een gewone spiegel, terwijl er vrijwel geen vervorming is. Helaas zal de nieuwe spiegel in de nabije toekomst nog weinig gebruikt worden: in de Verenigde Staten moet de buitenspiegel aan de bestuurderszijde plat zijn. *www.ams.org*

Masterplan Toekomst Wiskunde 2.0

Op verzoek van het ministerie van OCW is het advies ‘Masterplan Wiskunde 2.0’ opgesteld. Hoewel er positieve ontwikkelingen te zien zijn in de Nederlandse wiskunde, bijvoorbeeld de sterke groei van de studentenaantallen, is er reden tot zorg.

De opstellers van het advies, een werkgroep van Nederlandse topwiskundigen voorgezeten door prof.dr. Remco van der Hofstad, constateren dat de omvang van de wiskundige staf aan universiteiten achterblijft bij de groei van de studentaantallen. De verwachting is dat dit in de nabije toekomst problemen gaat opleveren bij bètastudies. De vorming van de wiskundeclusters, gefocust op gebieden waarin de Nederlandse wiskunde het best presteert, is succesvol, maar de financiering van deze clusters loopt eind 2013 af. Daarnaast wijzen de schrijvers op het belang van wiskunde voor het innovatievermogen van Nederland, en pleiten voor een langlopende investering in de Nederlandse wiskunde om dit innovatievermogen te behouden.

Het advies is op 6 juni overhandigd aan de Minister van OCW. *www.platformwiskunde.nl*

Slimmer netwerk door wiskundige technieken

Op het elektriciteitsnet moeten vraag en aanbod steeds op elkaar afgestemd zijn. In een net met een bescheiden aantal centrales is dit relatief eenvoudig te bereiken. Door de opkomst van duurzame energiewinning, bijvoorbeeld door zonnepanelen of windmolens, wordt energie steeds kleinschaliger opgewekt en varieert de opbrengst veel sterker. Hierdoor is het afstemmen van vraag en aanbod veel moeilijker geworden. Maurice Bosman, die op 5 juli promoveerde aan de Universiteit Twente, heeft onderzocht hoe het elektriciteitsnetwerk beter aangestuurd kan worden om dit te bereiken zonder dat energieleveranciers hoeven te investeren in te veel overcapaciteit. De methode TRIANA, ontwikkeld door het onderzoeksinstituut CTIT, blijkt hiervoor geschikt te zijn.

Bosman werd begeleid door prof.dr. Johann Hurink en prof.dr.ir. Gerard Smit. Het onderzoek werd ondersteund door STW, Essent en GasTerra. *www.utwente.nl*

Spinozapremie voor Ieke Moerdijk

Wiskundige Ieke Moerdijk is door NWO bekroond met een Spinozapremie, de hoogste onderscheiding in de Nederlandse Wetenschap. Moerdijk is hoogleraar algebra en topologie aan de Radboud Universiteit Nijmegen en is een van de grondleggers van de algebraïsche verzamelingenleer. Zijn meer recente werk gaat over algebraïsche topologie en differentiaalmeetkunde.

De Spinozapremie wordt jaarlijks toegekend aan maximaal vier excellente onderzoekers als eerbewijs aan hun wetenschappelijke carière. De premie, een bedrag van 2,5 miljoen euro, is vrij te besteden aan onderzoek naar keuze. *www.nwo.nl*

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An interview with Henri Poincaré

Mathematics is the art of giving the same name to different things

It looked like a daunting and perhaps even impossible enterprise to interview the famous professor Henri Poincaré. However, it turned out to be possible on the condition that professor Poincaré was permitted to formulate the questions himself. We were happy to accept this condition. The interview is elaborated by Ferdinand Verhulst.

[We start with some questions about the foundations of mathematical thinking.]
Question: What is the nature of mathematical reasoning? Is it really deductive as is usually believed? [1, essay 'Sur la nature de la raisonnement mathématique']
Answer: A deep analysis shows us that it is not, that it uses to a certain measure inductive reasoning and in this way it is fruitful. Opening an arbitrary book on mathematics, we find the author announcing that he wants to generalize a known proposition. So, the mathematical method proceeds from the particular to the general and how is it then that we can call it deductive?


Henri Poincaré receives visitors in his office at home

[Thinking about the foundations of mathematics can be interesting, but a question for the 'mathematician at work' is whether exploring the foundations actually improves our mathematical reasoning.]
Question: Have we achieved absolute rigour [in mathematics]? In each stage of the evolution, our predecessors believed to have achieved this. If they were wrong, are we not also wrong like them? [2, essay 'L'intuition et la logique en mathématiques']
Answer: We believe that in our reasoning, we don't need intuition. The philosophers tell us that this is an illusion. Pure logic will lead us only to tautologies. This cannot create anything new, from logic alone no science can emerge. For the other thing we need, we have no other word than 'intuition'. If one wants to take the trouble to be rigorous in todays' analysis, there are only syllogisms or an appeal to the intuition of numbers [induction], the on-

ly one which cannot deceive us. One can say that today absolute rigour has been achieved.

[You have been very productive in mathematics and in other fields, so a few questions about research strategy are of interest to us.]
Question: Pure analysis puts a great many procedures at our disposal with a guarantee for correct answers. But, from all the roads we can take, which one will lead us as quickly as possible to the goal? Who will tell us which one to choose? [2, essay 'L'intuition et la logique en mathématiques']
Answer: We need a faculty that shows us the goal from far away and this faculty is 'intuition'. Logic and intuition both play a necessary part. Both are indispensable. Logic can give certainty only and is the instrument of proof, intuition is the instrument of invention.

[What can we say about the relation between mathematics, which is concerned with objects of the mind, and the empirical sciences. Some people from the physical sciences will say "mathematics is just a tool, we use as little of it as possible, most mathematics is too artificial".]
Question: Experience is the only source of truth, only this can give us certainty. But if experience is everything, what place remains for mathematical physics? What has experimental physics to do with such a help which seems useless and perhaps even dangerous? [1, essay 'Les hypothèses en physique']
Answer: The scientist must order; one makes science with facts as one makes a house with stones. A big collection of facts is no more a science than a heap of stones is a house. The efforts of scientists have tended in resolving complex phenomena, arising directly from experiments, into a great many of elementary phenomena. The knowledge of elementary facts enables us to put the problems in equations.
Question: What is the objective value of science? And first, what should be understood by objectivity? [2, essay 'La science et la réalité']
Answer: The first condition of objectivity is, that what is objective must be considered as such by a number of spirits, and consequently must be able to be transmitted from one to the other. This transmission can only be done by 'discourse'. No discourse, no objectivity. A second condition is that scientific phenomena correspond with sensations that are actually tested. The first condition separates reality from a dream, the second one from a novel.

[In mathematics we study objects existing only in our mind, we mentioned this before. When we are aware of this, can we expect that mathematical physics leads to knowledge of reality?]
Question: Does science tell us the real nature of things? Does science tell us the real nature of the relation between things? [2, essay 'La science et la réalité']
Answer: About the first question nobody hesitates to answer 'no'. But I believe one can go further: not only can science not tell us about the nature of things, nothing is able to tell us that and if some god knew it, he could not find the words to express it. Not only could we not guess the answer, but if one would give it to us, we would not be able to understand it. I even ask myself whether we have a good understanding of the question.

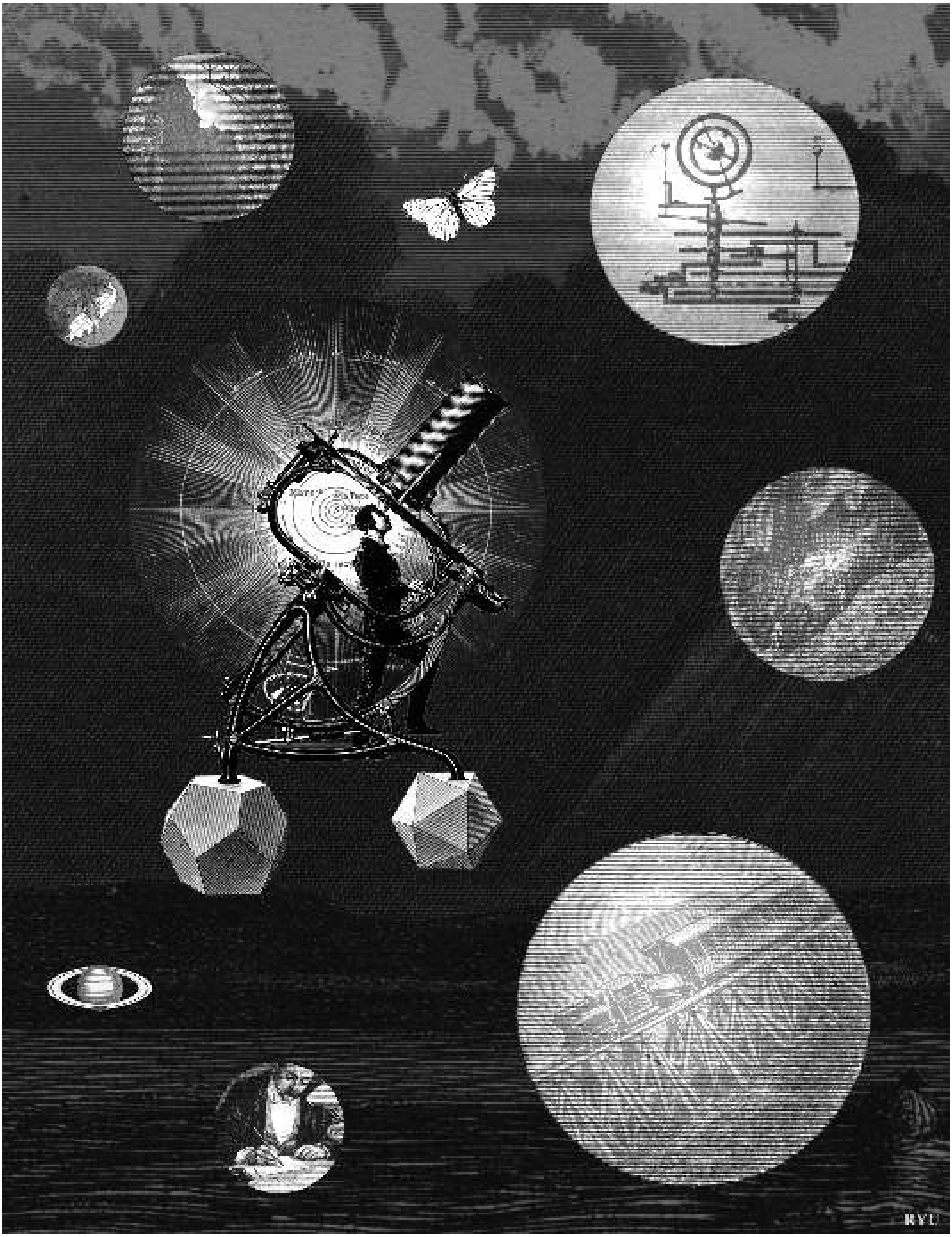
To understand the meaning of the second question we have to consider the conditions of objectivity. Are these relations the same for everybody? It is essential that everybody who is knowledgeable on experiments agrees about it. The question is then whether this agreement will persist with our successors. One will say that science is only a classification and that a classification can not be true but only convenient. It is true that it is convenient, but not only for me but for all men. It is true that it will remain convenient for our descendants, it is true finally that this can not be accidental.

[In science, we assume that the physical laws exist for all time and the question of the permanence of physical laws is usually avoided. Can we say something about this?]
Question: Were the laws of nature of former eras those of today? Will the laws of tomorrow still be the same? [4, essay 'L'évolution des lois']
Answer: If we imagine two minds similar to ours observing the universe on two occasions differing for example by millions of years, each of these minds will construct a science which will be a system of laws deduced from observed facts. It is probable that these sciences will be very different and in that sense it could be said that the laws have evolved. But however great the difference may be, it will always be possible to conceive of an intellect, of the same nature as ours but endowed with a much longer life, which will be able to complete the synthesis. To this intellect, the laws will not have changed, science will be unalterable; the scientists will merely have been imperfectly informed.

[Here is a very practical question that has to do with deterministic, but chaotic phenomena in nature. You were the first to write about this in your work on dynamical systems. Roughly 70 years after this, scientists became aware of these fundamental aspects of nature by the papers of Lorenz on an atmospheric model and by Hénon and Heiles on a galactic model. It is amazing that no scientist in physics or engineering picked this up before. The following questions and answers show your insight in the problem of the predictability of real-life phenomena.]
Question: Why have meteorologists such difficulty in predicting the weather with any certainty? [3, essay 'Le hasard']
Answer: We see that great disturbances are generally produced in regions where the atmosphere is in unstable equilibrium. The meteorologists see very well that the equilibrium is unstable, that a cyclone will be formed somewhere, but they are not in a position to say exactly where. A tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extends its ravages over districts it would otherwise have spared. Here, again, we find the same contrast between a very trifling cause that is inappreciable to the observer, and considerable effects, that are sometimes terrible disasters.
Question: Do these probability considerations apply outside science? [3, essay 'Le hasard']
Answer: It is the same in the humanities, and particularly in history. The historian is obliged to make a selection of the events in the period he is studying, and he only recounts those that seem to him the most important. Thus he contents himself with relating the most considerable events of the sixteenth century, for instance, and similarly the most remarkable facts of the seventeenth century. If the former are sufficient to explain the latter, we say that these latter conform to the laws of history. But if a great event of the seventeenth century owes its cause to a small fact of the sixteenth century that no history reports and that everyone has neglected, then we say that this event is due to chance, and so the word has the same sense as in the physical sciences; it means that small causes have produced great effects.

The greatest chance is the birth of a great man. No example can give a better understanding of the true character of chance.

[There is a special research topic that touches upon the fundamentals of mathematics, the understanding of the continuum of the real



Illustratie: Ryu Tajiri

numbers. This topic was important in your time but it is still important.]

Question: What is exactly this continuum, this subject of mathematical reasoning? [1, essay 'La grandeur mathématique et l'expérience']

Answer: The continuum is nothing else than a set of elements, sequentially arranged, certainly infinite, but separated from each other. Our definition is however not complete. [...] One can ask whether the concept of mathematical continuum has not simply been derived from experience. One is forced to conclude that this idea has been created completely by the human spirit, but that experience has induced it.

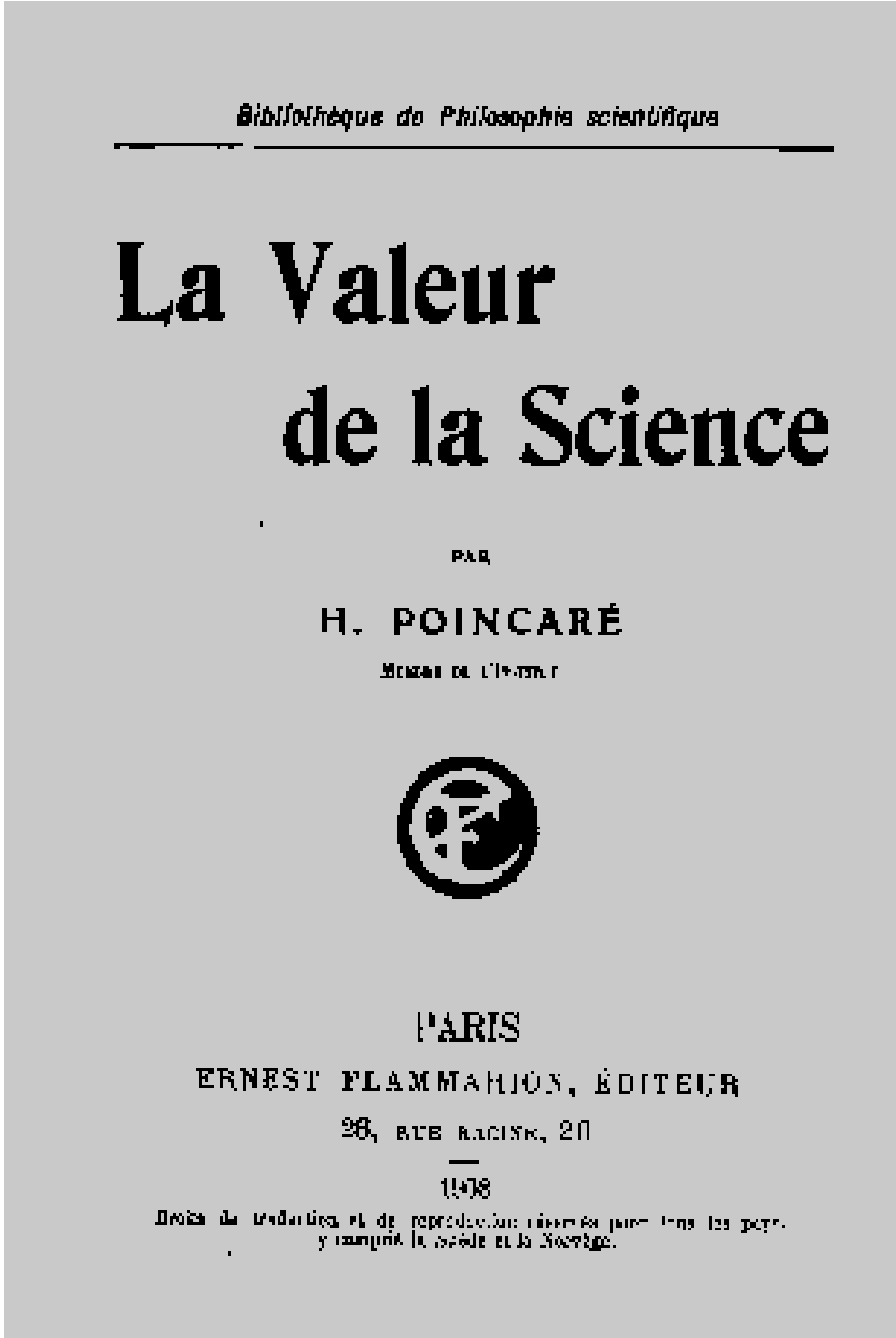
Question: Is the creative power of the mind regarding the mathematical continuum exhausted? [1, essay 'La grandeur mathématique et l'expérience']

Answer: No, the work of Du Bois-Reymond shows this in a remarkable way. One knows that mathematicians distinguish between infinitesimal small [quantities] of different orders and that those of second order are not only infinitesimal small in an absolute sense, but also with respect to those of first order. It is not difficult to imagine infinitesimal small quantities of fractional or even irrational order and in this way we find again an ordering of the mathematical continuum.

[The styles and methods of proofs in mathematics have evolved since your time. With the enormous growth of both pure and applied mathematics there is now an abundance of mathematical styles, there are even computer-assisted proofs. Does it make sense to distinguish between beautiful and ugly mathematics, between elegant and graceless reasoning?]

Question: Mathematicians attach a great importance to the elegance of their methods and of their results, and this is not mere dilettantism. What is it that gives us the feeling of elegance in a solution or proof? [3, essay 'L'avenir des mathématiques']

Answer: It is the harmony of the different parts, their symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts. Elegance may result from the feeling of surprise caused by the unlooked-for occurrence of objects not habitually associated. In this, again, it is fruitful, since it discloses thus relations that were until then unrecognized. Mathematics is the art of giving the same names to different things.


Bibliothèque de Philosophie scientifique

La Valeur de la Science

PAR

H. POINCARÉ

Membre de l'Institut

PARIS

ERNEST FLAMMARION, ÉDITEUR

26, RUE RACINE, 26

[illegible]

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[In your time, you were one of the last universal scientists, but then, scientists started to specialize in their research. Is this a danger or a necessity, or both?]

Question: How should one view specialization? [3, essay 'L'avenir des mathématiques']

Answer: As science develops, it becomes relatively more difficult to grasp it in its entirety. Then an attempt is made to cut it in pieces and to be satisfied with one of these pieces — in one word, to specialize. Too great a movement in this direction would constitute a serious obstacle to the progress of science. As I have said, it is by unexpected concurrences between its different parts that it can make progress. Too much specialization would prohibit these concurrences.

[You wrote many textbooks, but why are some people afraid of mathematics? To prove and invent new mathematics, one needs intuition, but to understand school mathematics one needs only some natural logic.]

Question: How is it that there are so many minds that are incapable of understanding

[even elementary] mathematics? Here is a science which appeals only to the fundamental principles of logic; is there not something paradoxical in this? [3, essay 'Les définitions mathématiques et l'enseignement']
Answer: What is understanding? Has the word the same meaning for everybody? Does understanding the proof of a theorem consist in examining each of the syllogisms of which it is composed in succession, and being convinced that it is correct and conforms to the rules of the game? Not for the majority [of people]. They want to know not only whether all the syllogisms of a proof are correct, but why they are linked together in one order rather than in another. As long as they appear to be developed by caprice, and not by an intelligence constantly conscious of the end to be attained, they do not think they have understood. At first they still perceive the evidence that is placed before their eyes, but, as they are connected by too attenuated a thread with those that precede and those that follow, they pass without leaving a trace in their brains, and are immediately forgotten.

[In teaching we would like to convey mathematical ideas to students, but the problem is that often we have to adapt to their needs. Is there a fixed mathematics curriculum for all?]
Question: The engineer must receive a complete mathematical training, but of what use is it to him, except to enable him to see the different aspects of things and to see them quickly? [3, essay 'Les définitions mathématiques et l'enseignement']
Answer: He has no time to split hairs. In the complex physical objects that present themselves to him, he must promptly recognize the point where he can apply the mathematical instruments we have put in his hands. The principal aim of mathematical education is to develop certain faculties of the mind, and among these intuition is not the least precious. It is through it that the mathematical world remains in touch with the real world. The practitioner will always need it, and for every pure geometrician there must be a hundred practitioners.

[You have been active in science but also in reaching out to the general public in lectures and essays. Is it not difficult that so many people are doubtful about the value of mathematics?]
Question: People often ask what the use is of mathematics and if these delicate constructions that emerge completely out of our mind are not artificial and created by a whim? [2, essay 'L'analyse et la physique']
Answer: Among the people who ask this question, I make a distinction. Some down-to-earth people are asking only from us a way to make money; those people do not merit an answer. It would be better to ask those who spend their time to become rich, what good it is to neglect at the same time art and science, the only things that enable the souls to enjoy themselves. By the way, science that is only concerned with applications, is impossible; results are only productive when they are connected to each other.

Other people are interested only in nature and they ask us if we are able to improve our knowledge of it. Mathematics has three purposes. It delivers an instrument for the study of nature, but this is not all. It has a philosophical purpose and, I dare say, an aesthetic purpose. These purposes can not be separated and the best way to achieve one purpose is to aim at the other ones, or at least not to loose sight of them.

[Funding of science is an issue that remains opportune at all times.]
Question: Governments and parliaments find astronomy an expensive science; is this correct? [2, essay 'L'astronomie']
Answer: Politicians should have kept a certain rudimentary idealism. Astronomy is useful because it lifts us above ourselves, it is useful because it is great, it is useful because it is beautiful. Astronomy has facilitated the results of other, more directly useful sciences because it has made us capable of understanding nature. The success of astronomy has been that nature follows laws, these laws are unavoidable, one can not ignore them.

[Does funding of science imply a specific political choice of the scientists?]
Question: Should scientists with political interest fight or support the government? [In periodical *La Revue Bleue*, 1904]
Answer: Well, this time I have to excuse myself; everybody will have to choose according to his conscience. I think that not everybody will cast the same vote, and I see no reason to complain about this. If scientists take part in politics, they should take part in all parties, and it is indeed necessary that they be present in the strongest party. Science needs money, and it should not be such that the people with power can say, science, that is the enemy.

Here ends our interview with the amazingly creative professor Poincaré. He was aware of the extent and the importance of his own achievements, but this did not make him rest on his laurels. He never stopped questioning results in the mathematical sciences, their foundations and their role in society until the end of his life.

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Henri Poincaré and partial differential equations

Henri Poincaré introduced a new approach to solve the Dirichlet problem and he gave the first general solution of the initial value problem for the 'telegraph equation'. In this article Jean Mawhin describes these and other contributions of Poincaré to partial differential equations.

Henri Poincaré has been professor of mathematical physics and probabilities from 1886 until 1896, when he exchanged this chair for that of theoretical astronomy and celestial mechanics left vacant by Félix Tisserand's unexpected death. The story of the surprising nomination of a specialist in number theory and analysis at this chair of mathematical physics is told in [2]. All Poincaré's papers about partial differential equations have been published after 1886. Many of his original contributions or presentations are contained in the well-known series of textbooks based upon his lectures at the Sorbonne.

Between 1890 and 1896, Poincaré devoted three long and important *memoirs* [26, 29, 33] to the partial differential equations of mathematical physics. Among the achievements in those papers are the first general existence result for the Dirichlet problem associated to the Laplacian on a general bounded domain, an implicit minimax characterization of its eigenvalues, an important inequality and the first existence proof of all eigenvalues for the same problem.

Other significant contributions of Poincaré to partial differential equations deal with the first general solution of the telegraph equation on an infinite line in 1893, and the use of a continuation method for the solution of some non-linear elliptic problems. We briefly analyse them in this paper but, to keep it within a reasonable size, we have excluded other Poincaré contributions to partial differential equations, like his papers dealing with the propagation and diffraction of Hertzian waves.

Dirichlet problem and sweeping out method

Given a domain $D \subset \mathbb{R}^3$ bounded by a surface S, and a continuous function Φ over S, a *Dirichlet problem* consists in finding a function V such that

$$\begin{aligned} &\Delta V := \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{in } D, \\ &V = \Phi \quad \text{on } S. \end{aligned} \tag{1}$$

If $\nabla U := (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z})$, any minimizer V of the integral $\mathcal{I}(U) := \int_D \|\nabla U\|^2$ over all functions $U \in C^2(D) \cap C(\overline{D})$ equal to Φ on S is a solution of (1). Bernhard Riemann called the argument *Dirichlet's principle*, used before him by Johann Peter Lejeune Dirichlet and William Thomson, claiming the existence of a minimum from the positivity of $\mathcal{I}(U)$. After Karl Weierstrass' criticism, several mathematicians of the end of the nineteenth century, like Carl Neumann and Gustave Robin, searched for other existence proofs for the Dirichlet problem, which, according to Poincaré, "are methods of proof of a solution of the problem and computational methods to solve it effectively. As methods of proof, they are rather complicated, but they complete each other in order to apply to all cases and to satisfy the most severe judges with respect to rigour. As methods of computation, they have no value; even the sim-

Johann Dirichlet (1805–1859)

plest of them [...] leads to inextricable computations from the second approximation."

In [26], Poincaré introduced a new approach to solve the Dirichlet problem in bounded domains $D \subset \mathbb{R}^3$ with sufficiently smooth boundary S. He gave a modest and lucid appreciation of his method: "The most interesting thing would have been to replace the present methods of computations by less defective ones. I was unable to do it, and restricted myself to find a method of proof simpler than the ones proposed until now, and directly applicable to all cases."

The main ideas of his *sweeping out method* were announced on 3 January 1887 in a note of the *Comptes Rendus* [23], and developed in [26] for the exterior Dirichlet problem. Poincaré applied it directly to the Dirichlet problem (1) in his book [34], and we describe it here in this more familiar frame. The *sweeping out of a sphere S* consists in replacing matter (or charge) inside of S by an equivalent distribution of matter (or charge) on S. In this way, a material point M of mass one can be replaced by a mass one distributed on the boundary with a superficial density inversely proportional to the cube of the distance to M, and the result is easily extended to an arbitrary mass distribution inside S. This process does not introduce any negative mass, does not modify the potential outside S and diminishes it inside S.

Approximating Φ through a series of polynomials on a closed ball B containing D in its interior, and using the maximum principle and Harnack's theorem, Poincaré showed that *it suffices to solve the Dirichlet problem for data which are the restriction to S of a polynomial P*. Poincaré first expressed D as the union of a sequence of balls (B_n) contained in D, and assumed that $\Delta P < 0$ on B. Letting $\Delta P = -4\pi\sigma$ and $W_0 = \int_B \frac{\sigma}{r}\, dx$, in such a way that $\Delta W_0 = -4\pi\sigma = \Delta P$ in B, and looking at σ as a density of attracting matter, Poincaré swept out the balls B_n in the order

$$B_1, B_2; B_1, B_2, B_3; B_1, B_2, B_3, B_4; \ldots,$$

so that every ball is swept out an infinity of times. If W_n denotes the potential of the attracting matter after the nth operation, it is clear that $W_n > 0$ and that $W_n \le W_{n-1}$. Consequently, the sequence (W_n) converges pointwise, say to W, with $0 < W < W_0$ in D and $W = W_0$ outside of D. If the ball B_k has been swept out during the operations numbered by $\alpha_1, \alpha_2, \alpha_3, \ldots$, the functions W_{α_j} are harmonic in B_k, and the sequence (W_{α_j}) converges to W. From Harnack's theorem, W is harmonic in B_k, and hence in $D = \cup_{k\ge1} B_k$.

Poincaré then showed that *W is continuous at any point Q of S in which S has a tangent plane and two principal curvature radii*. The function $V = W - W_0 + P$ is harmonic in D since $\Delta W = 0$ and $\Delta W_0 = \Delta P$, and is equal to P on S, since $W = W_0$ on S. The Dirichlet problem is therefore solved when $\Delta P < 0$ and S has the mentioned regularity at each point. The restriction $\Delta P < 0$ can be easily removed because any polynomial P can be written as the difference $P_2 - P_1$ of two polynomials such that $\Delta P_1 < 0$ and $\Delta P_2 < 0$. If V_j is the harmonic function in D equal to P_j on S $(j = 1, 2)$, then $V = V_1 - V_2$ is the searched function.

The regularity conditions upon the boundary have been generalized by Henri Lebesgue and by Norbert Wiener. A short account of Poincaré's work is given in [6]. More details can be found in [16] and [40]. It is superfluous to insist over the consequences and developments of the sweeping out method in potential theory and in the study of elliptic equations, in the hands of mathematicians like Stanislaus Zaremba, Henri Lebesgue, Oskar Perron, Norbert Wiener, Frédéric Riesz, Otto Frostman, Charles de La Vallée Poussin, Henri Cartan, Marcel Brelot and others [5, 19].

Variational characterization of the eigenvalues of Fourier's problem

Separation of space and time dependence in *Fourier's problem for heat equation* leads to the determination of positive numbers k and functions U such that

$$\Delta U + kU = 0 \quad \text{in } D \subset \mathbb{R}^3,$$
$$\frac{\partial U}{\partial n} + hU = 0 \quad \text{on } S, \quad \int_D U^2 = 1,$$

with $D \subset \mathbb{R}^3$ a bounded domain with boundary S, $\frac{\partial U}{\partial n}$ the exterior normal derivative, and h a positive constant. For special domains D, the problem can be explicitly solved, but for an arbitrary bounded D, Poincaré observed that "...the first point is to establish the existence of those functions U. This has not been done yet, as far as I know, in the general case, and I will try to do it."

The first approach to this problem, also contained in [26], was announced in two notes of the *Comptes Rendus* from 20 June 1887 [24] and 17 December 1888 [25]. Poincaré was aware of their lack of rigour, but not of their lack of novelty. A previous work of Heinrich Weber in 1869 [43] had proposed an entirely similar approach in dimension two. The underlying idea of both works is an extension of Dirichlet's principle. If $B(U_1)$ denotes the minimum of the positive quadratic expression

$$B(F) := h \int_S F^2 d\sigma + \int_D \|\nabla F\|^2$$

on $\Sigma_1 := \{F \in C^1 : \int_D F^2 = 1\}$, there is a Lagrange multiplier k_1 such that

$$\Delta U_1 + k_1 U_1 = 0 \quad \text{in } D \subset \mathbb{R}^3,$$
$$\frac{\partial U_1}{\partial n} + hU_1 = 0 \quad \text{on } S.$$

The existence of a *fundamental function* U_1, in Poincaré's terminology, is 'proved' in this way, and Poincaré observed that

$$k_1 \leq \frac{h\int_S F^2 d\sigma + \int_D \|\nabla F\|^2}{\int_D F^2},$$

for all non-zero functions F.

Similarly, if $B(U_2)$ denotes the minimum of $B(F)$ on $\Sigma_2 \subset \Sigma_1$ defined by $\Sigma_2 := \{F \in C^1 : \int_D F^2 = 1,\ \int_D FU_1 = 0\}$, there are Lagrange multipliers k_2 and λ such that

$$\Delta U_2 + k_1 U_2 + \lambda U_2 = 0 \quad \text{in } D \subset \mathbb{R}^3,$$
$$\frac{\partial U_2}{\partial n} + hU_2 = 0 \quad \text{on } S.$$

The conditions $\int_D U_1^2 = 1$, $\int_D U_1 U_2 = 0$ and Green's formulas imply that $\lambda = 0$, and hence U_2 is a second fundamental function. If we now minimize F on $\Sigma_3 = \{F \in C^1 : \int_D F^2 = 1,\ \int_D FU_1 = 0,\ \int_D FU_2 = 0\}$, we obtain the existence of some number k_3 and of a third fundamental function U_3, and, going on in this way, the existence of infinitely many fundamental functions. Poincaré admitted the non-rigorous character of his reasoning: "This proof is completely analogous to the one used by Riemann to establish Dirichlet's principle, that analysts have since replaced by more rigorous reasoning."

Poincaré was one of them, with his sweeping out method. Poincaré went further than Weber in obtaining an *upper bound* for k_n. Given $F_1, \ldots, F_n$, if $F = \sum_{j=1}^n \alpha_j F_j$, then $B(F)$ and $A(F) := \int_D F^2$ are positive definite quadratic forms in $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a simple algebraic reasoning shows that the roots $\lambda_1 \leq \cdots \leq \lambda_n$ of the characteristic equation $|B(F) - \lambda A(F)| = 0$ verify the inequalities $k_j \leq \lambda_j$ $(1 \leq j \leq n)$. As George Polya noticed later [39], if one takes $F_j = U_j$ $(1 \leq j \leq n)$, then $k_j = \lambda_j$ $(1 \leq j \leq n)$, and, if $S_n = \{\sum_{j=1}^n \alpha_j F_j : \alpha \in \mathbb{R}^n\}$, then $\lambda_n = \max_{F \in S_n \setminus \{0\}} B(F)/A(F)$. Consequently, Poincaré's reasoning *implicitly* contained the *minimax characterization of* k_n,

$$k_n = \min_{S_n} \max_{F \in S_n} \frac{B(F)}{A(F)},$$

explicitly given for the first time in 1905, for a finite-dimensional spectral problem, by Ernst Fisher [8], without any reference and through an algebraic approach independent of the calculus of variations. It is not necessary to insist on the importance and the development of minimax methods for the study of eigenvalues, by Herman Weyl, Richard Courant, Alexander Weinstein, George Polya, Menahem Schiffer and many others [10, 44].

Using once more Green's formula, Poincaré showed the increasing character of the k_j with respect to h. As, by construction, they also increase with j, it suffices to treat the case where $h = 0$, i.e. *Neumann's problem,* to show that $k_j \to +\infty$ as $j \to +\infty$. To this aim, Poincaré searched a *lower bound* for k_n. Decomposing D in p parts D_i $(1 \leq i \leq p)$, calling $U_{i,j}$ and $k_{i,j}$ the fundamental functions and the characteristic numbers associated to Neumann conditions on D_i, and letting $V = \sum_{l=1}^n \alpha_l U_l$, where the U_l are the first n characteristic functions on D and the α_l undetermined coefficients, it is easy to see that

$$\frac{\int_D \|\nabla V\|^2}{\int_D V^2} = \frac{\sum_{l=1}^n k_l \alpha_l^2}{\sum_{l=1}^n \alpha_l^2} \leq k_n. \tag{2}$$

Choosing the coefficients in such a way that $\int_{D_i} VU_{i,1} = 0$, $j = 1, \ldots, \lambda_i$, $i = 1, \ldots, n-1$, Poincaré obtained easily that

$$\int_D \|\nabla V\|^2 = \sum_{i=1}^{n-1} \int_{D_i} \|\nabla V\|^2 \geq \min_{1 \leq i \leq n-1} \{k_{i,2}\} \int_D V^2,$$

and hence, using (2), that $k_n \geq \min_{1 \leq i \leq n-1}\{k_{i,2}\}$. In particular, a polyhedron bounded by faces parallel to one of the planes of coordinates can be decomposed into $n-1$ rectangular parallelotopes whose largest side tends to zero when $n \to \infty$. If those largest sides are all smaller or equal to a_n, one obtains, by separation of variables, $k_{i,2} \geq \pi^2/a_n^2$ $(i = 1, \cdots, n-1)$, and hence $k_n \geq \pi^2/a_n^2$, so that $k_n \to +\infty$ when $n \to \infty$. If one can approach D by such polyedra the result remains valid for D. But Poincaré did not content himself with this reasoning and proposed to find a lower bound to k_2 when D is a bounded convex domain. This is the origin of the famous *Poincaré inequality*, to which the next section is devoted.

Poincaré inequality

Let $D \subset \mathbb{R}^3$ be a bounded and convex open set. Poincaré first observed that, with $\mu(D)$ the volume of D,

$$\int_{D \times D} [V(x) - V(x')]^2\, dx\, dx' = 2\mu(D) \int_D V^2 - 2\left(\int_D V\right)^2.$$

Therefore

$$\begin{aligned} k_2 &= 2\mu(D) \min_{V \neq 0: \int_D V = 0} \frac{\int_D \|\nabla V\|^2}{\int_{D\times D}[V(x) - V(x')]^2\, dx\, dx'} \\ &= 2\mu(D) \min_{V \neq 0} \frac{\int_D \|\nabla V\|^2}{\int_{D\times D}[V(x) - V(x')]^2\, dx\, dx'}, \end{aligned} \tag{3}$$

as the minimized expression does not change by adding a constant to V. Using a complicated argument based upon a spherical type change of variable, on which we return later, and upon a reduction to a one-dimensional problem of the calculus of variations, Poincaré obtained, *for continuously differentiable functions V such that* $\int_D V = 0$, the inequality

$$k_2 \geq \frac{6K_0 \mu(D)}{\pi d^5}, \tag{4}$$

where K_0 is some numerical constant and d the diameter of D.

This first formulation of the *Poincaré inequality* allowed him to extend his result $\lim_{n \to \infty} k_n = +\infty$ to the case of a 'general' domain. If one decomposes D in $n-1$ convex parts D_j $(1 \leq j \leq n-1)$, and if $\mu(D_j)/d_j^5 \geq \alpha$ for any D_j and some $\alpha > 0$, one easily gets $k_n \geq 6K_0\alpha/\pi$. The quantity α (of the order of d_j^{-2}) can be taken arbitrarily large by taking n sufficiently large and choosing suitably the parts D_j. Hence, for such a domain D, $k_n \to \infty$ when $n \to \infty$.

Poincaré returned to inequality (4) in the third section 'Preliminary lemma' of his second memoir [29] on the equations of mathematical physics. He substantially simplified his argument and obtained a more explicit expression than in (4). For simplicity, we restrict to the case of a bounded convex open set D of $\mathbb{R}^2$, for which Poincaré only stated the result. To estimate the right-hand member of (3), Poincaré introduced, like in the preceding memoir, the change of variables

$$\begin{aligned} x &= \xi + \rho\cos\varphi, \quad y = \rho\sin\varphi, \\ x' &= \xi + \rho'\cos\varphi, \quad y' = \rho\sin\varphi. \end{aligned} \tag{5}$$

Geometrically, ξ is the first component of the intersection of the line joining (x, y) to (x', y') with the first axis, φ the angle of the segment joining $(\xi, 0)$ to (x, y) with this axis, ρ the distance of (x, y) to $(\xi, 0)$ and ρ' the distance of (x', y') to $(\xi, 0)$. The aim of this change of variables is essentially to reduce the inequality to the one-dimensional case. Letting $T(\xi, \varphi, \sigma) = (\xi + \sigma\cos\varphi, \sigma\sin\varphi)$, and using the Cauchy inequality, Poincaré obtained the estimate

$$\left[\frac{\partial}{\partial\rho}(V\circ T)\right]^2 \le \left(\frac{\partial V}{\partial x}\circ T\right)^2 + \left(\frac{\partial V}{\partial y}\circ T\right)^2 = \|\nabla V\circ T\|^2. \tag{6}$$

If $\mu(D)$ denotes the area of D, using (6), one has

$$\mu(D)\int_D \|\nabla V\|^2 = \int_{D\times D} \|\nabla V(x)\|^2\, dx\, dx'. \tag{7}$$

The idea consists in using the transformation (5) to compute the right-hand member integral as well as $\int_{D\times D}[V(x) - V(x')]^2\, dx\, dx'$, to notice that, for $\rho' > \rho$, the Schwarz inequality implies, with $T' = T(\xi, \varphi, \sigma')$,

$$(V\circ T - V\circ T')^2 \le (\rho' - \rho)\int_\rho^{\rho'} \left[\frac{\partial}{\partial\sigma}(V\circ T)\right]^2 d\sigma,$$

integrate this inequality over the other variables and use (6) to obtain the *Poincaré inequality in the plane*

$$\int_D \|\nabla V\|^2 \ge \frac{24}{7d^2}\int_D V^2, \tag{8}$$

for any continuously differentiable function V such that $\int_D V = 0$.

When D is a bounded convex open subset of, a similar approach gives the *Poincaré inequality in space*

$$\int_D \|\nabla V\|^2 \ge \frac{16}{9d^2}\int_D V^2 \tag{9}$$

for all continuously differentiable functions V such that $\int_D V = 0$. Modern treatments of the Poincaré inequality only prove, using a compactness argument, the *existence* of a constant $c > 0$ such that

$$\int_D \|\nabla V\|^2 \ge c\int_D V^2 \tag{10}$$

when $\int_D V = 0$, without information about the dependence of c with respect to D. A recent work [3] shows that the best constant c for a convex $D \subset \mathbb{R}^n$ is equal to $\frac{\pi^2}{d^2}$ for all n. On the other hand, the inequality (10) for the functions *which vanish on S*, often referred as the Poincaré inequality in the literature, does not occur in the work of the French mathematician, but can be traced to Herman A. Schwarz.

Poincaré uses inequalities (8)–(9) to show that, given p functions $\varphi_1, \ldots, \varphi_p$ and $V = \sum_{i=1}^p \alpha_i\varphi_i$, one can choose the α_i in such a way that $\int_D \|\nabla V\|^2/\int_D V^2$ is larger than any number given in advance. First, if D can be decomposed into $p-1$ convex open sets D_i of diameters smaller than d, choosing the α_i in order that $\int_{D_i} V = 0 \quad (1 \le i \le p-1)$, inequality (9) applied to D_i implies that

$$\int_D \|\nabla V\|^2 = \sum_{i=1}^{p-1}\int_{D_i}\|\nabla V\|^2 \ge \frac{16}{9d^2}\sum_{i=1}^{p-1}\int_{D_i} V^2 = \frac{16}{9d^2}\int_D V^2.$$

If now D *is convex* and contained in a cube of side Λ, one divides this cube into q^3 equal cubes of side Λ/q. The intersection of such a small cube with D is convex and has diameter smaller than $\Lambda\sqrt{3}/q$. If one has $p-1$ intersections, the reasoning above shows that one can choose the α_i in such a way that

$$\int_D \|\nabla V\|^2 \ge \frac{16q^2}{27\Lambda^2}\int_D V^2.$$

It will be possible to do it *a fortiori* if p is larger than the number of intersections plus one, and it suffices to take $p \ge q^3 + 1$. Consequently, *for D convex, one can choose the α_i in such a way that*

$$\int_D \|\nabla V\|^2 \ge L_p\int_D V^2 \tag{11}$$

with $L_p = \frac{16q^2}{27\Lambda^2}$ and q^3 the largest perfect cube contained in $p-1$. If D can be decomposed into m convex sets, one still obtains the inequality (11) for a suitable choice of the α_i, with q^3 the largest perfect cube contained in $\frac{p-1}{m}$. Similar arguments work in dimension 2, and, when $p \to \infty$, $L_p \simeq p^{2/3}$ if $n = 3$ and $L_p \simeq p$ if $n = 2$.

More details on Poincaré's contributions to his inequality can be found in [1] and [16].

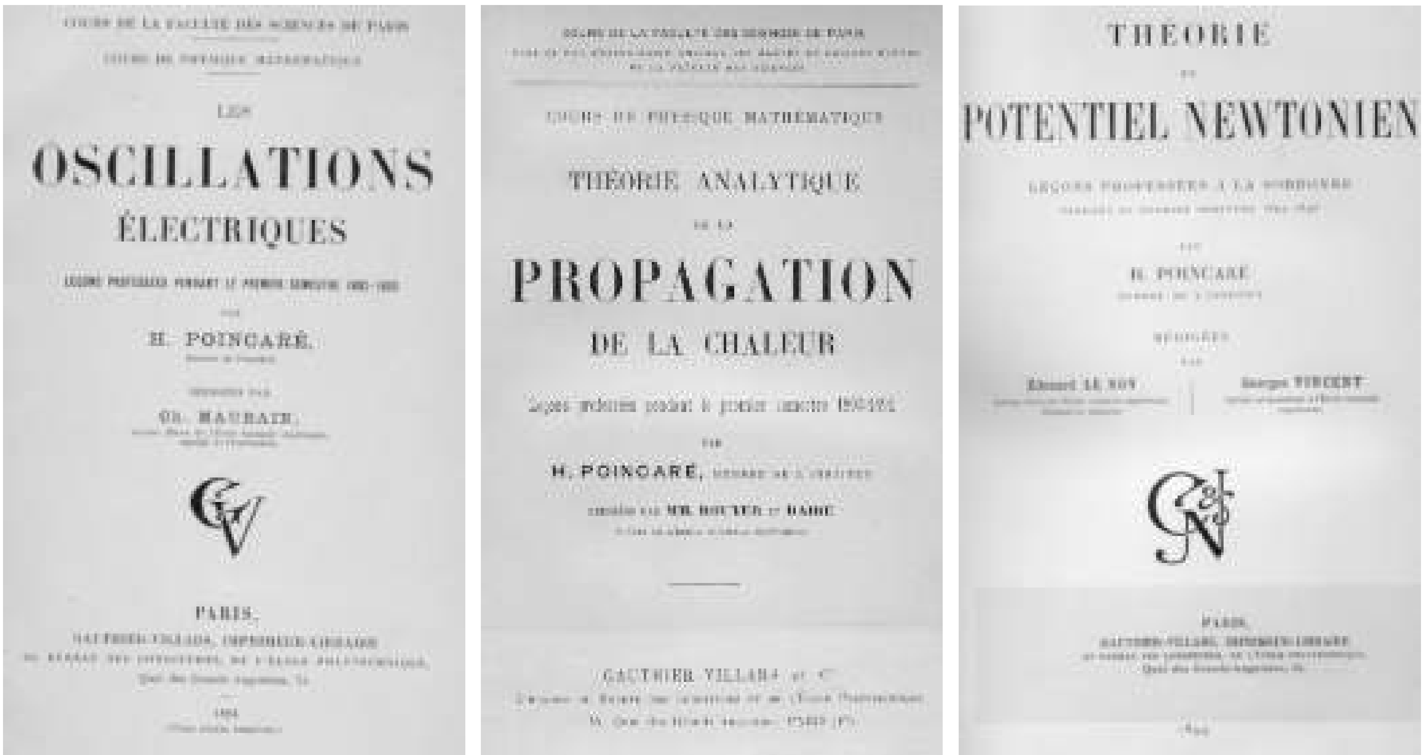
Eigenvalues and eigenfunctions of the Dirichlet problem

One of the main contributions of the memoir [29] of 1894 is to provide the first *rigorous* proof of the existence of eigenvalues and eigenfunctions for the Dirichlet problem in an arbitrary bounded domain. The existence of the first eigenvalue had been shown in 1885 by Herman A. Schwarz [41] and that of the second one in 1893 by Émile Picard [22]. Poincaré's result was announced in a note in the *Comptes Rendus* of 26 February 1894 [28].

Given a bounded domain $D \subset \mathbb{R}^3$ with boundary S, a continuous function f over D, and $\xi \in \mathbb{R}$, Poincaré's idea was to start by the study of the *non-homogeneous* Dirichlet problem

$$\Delta v + \xi v + f = 0 \quad \text{in } D, \quad v = 0 \quad \text{on } S, \tag{12}$$

and to detect the eigenvalues as the values of ξ for which the solution of (12) is infinite, i.e. for which *resonance* occurs. Following a method of Schwarz, he searched the solution in the form of a series $v = \sum_{k=0}^\infty \xi^k v_k$, leading formally to the sequence of Dirichlet problems



Front covers of some of Poincaré's publications [30–31, 34]

for the Laplacian

$$\begin{aligned} \Delta v_0 + f = 0 \quad &\text{in } D, \quad v_0 = 0 \quad \text{on } S, \\ \Delta v_k + v_{k-1} = 0 \quad &\text{in } D, \quad v_k = 0 \quad \text{on } S \ (k = 1, 2, \ldots), \end{aligned}$$

each of which having a unique solution by the sweeping out method. Using the 'Schwarz' integrals'

$$W_{m,n} := \int_D v_m v_n, \quad V_{m,n} := \int_D \nabla v_m \cdot \nabla v_n (m, n = 0, 1, 2, \ldots),$$

which are such that $W_{m,n} = W_{m+n,0} := W_{m+n}$, Poincaré showed that the sequence (W_{n+1}/W_n) converges to some $1/R$ such that $R \geq Q\sqrt{\mu(D)}$. This implies the uniform convergence (with respect to x) of the series $\sum_{n=0}^{\infty} \xi^n v_n(x)$ for $|\xi| < R$, and that *problem (12) has a solution $v = [f, \xi]$ for any $|\xi| < R$, and in particular for any $|\xi| < Q\sqrt{\mu(D)}$.*

If φ_j $(1 \leq j \leq p)$ are p given functions, $w_j = [\varphi_j, \xi]$ $(1 \leq j \leq p)$, $\varphi = \sum_{j=1}^{p} \alpha_j \varphi_j$, $v = \sum_{j=1}^{p} \alpha_j w_j$, then, by linearity, $v = [\varphi, \xi] = \sum_{n=0}^{\infty} \xi^n v_n$. Using an argument of nested closed sets, Poincaré showed that the α_j can be chosen in such a way that the corresponding R is larger or equal to L_p, where L_p is given by (11). To apply this observation to problem (12), let

$$v = [f, \xi] = \sum_{n=0}^{\infty} \xi^n v_n, \quad u_j = [v_{j-2}, \xi] \quad (2 \leq j \leq p).$$

If $u_j = \sum_{n=0}^{\infty} \xi^n u_{j,n}$, one has, with the zero Dirichlet condition on the boundary,

$$\Delta u_{j,0} + v_{j-2} = 0, \Delta u_{j,1} + u_{j,0} = 0, \ldots, \Delta u_{j,n} + u_{j,n-1} = 0, \ldots$$

and hence, by uniqueness of Dirichlet problem,

$$u_{j,0} = v_{j-1}, u_{j,1} = v_j, \ldots, u_{j,n} = v_{j+n-1}, \ldots$$

i.e. $u_j = \sum_{n=0}^{\infty} \xi^n v_{j+n-1}$ $(2 \leq j \leq p)$. Let now $w = \alpha_1 v + \sum_{j=2}^{p} \alpha_j u_j$. By linearity,

$$\begin{aligned} w = \sum_{n=0}^{\infty} \xi^n w_n &= \left[\alpha_1 v + \sum_{j=2}^{p} \alpha_j u_j, \xi \right] \\ &= \sum_{n=0}^{\infty} \xi^n \left(\alpha_1 v_n + \sum_{j=2}^{p} \alpha_j v_{j+n-1} \right). \end{aligned} \tag{13}$$

Using the result above, one can choose the α_j in such a way that the series (13) has a radius of convergence at least equal to L_p. Identifying the coefficients of the two power series for w and using Cramer's rule, one gets the expression $v = P(x, y, z, \xi)/D(\xi)$, with $D(\xi) = \sum_{j=0}^{p-1} (-1)^j \alpha_{p-j} \xi^j$ and $P(x, y, z, \xi)$

$$= \begin{vmatrix} w(x,y,z,\xi) & \alpha_2 & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p \\ v_0(x,y,z) & -\xi & 0 & \cdots & 0 & 0 \\ v_1(x,y,z) & 1 & -\xi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ v_{p-2}(x,y,z) & 0 & 0 & \cdots & 1 & -\xi \end{vmatrix}.$$

As $v_0, v_1, \ldots, v_{p-2}$ do not depend upon ξ and w is holomorphic in ξ for $|\xi| < L_p$, the function $v = P/D$ is meromorphic in ξ for $|\xi| < L_p$. As we can take p as large as we want, *v is meromorphic in ξ in the whole complex plane.* It is not difficult to see that $P(\cdot, \xi)$ vanishes on S, has partial derivatives of the first and second order with respect to x, y, z, and satisfies the partial differential equation

$$\Delta P = \Delta(vD) = D\Delta v = -\xi Dv + Df = -\xi P + Df. \tag{14}$$

The poles of modulus smaller or equal to L_p of the meromorphic function v are the zeros of the polynomial D. If $\xi = k$ is a simple zero of D, and $P_k = P(\cdot, k)$, then (14) implies $\Delta P_k + kP_k = 0$, which shows that k is a *characteristic number* associated to the *harmonic function* P_k, i.e., in present terminology, P_k is an *eigenfunction* of $-\Delta$ with Dirichlet conditions upon S, associated to the *eigenvalue* k. Poincaré justified as follows his obsolete terminology: "The various simple sounds that a membrane can produce are characterized by equations of the form $\Delta u + ku = 0$, the function being requested to vanish at the boundary. It is well known that those simple sounds have been called harmonics."

Poincaré showed then that v *only admits simple poles,* and hence, up to a multiplicative constant, P_k *is the residue of* v *at* k. Through relatively elementary considerations, he also showed that v cannot be holomorphic in the whole plane, which insures the existence of harmonic functions, that one cannot have more than $p-1$ linearly independent harmonic functions linearly with characteristic number smaller than L_p, and hence that *to any characteristic number can correspond only a finite number of linearly independent harmonic functions,* and finally that *all the characteristic numbers are real and positive.* Poincaré *normed* the harmonic functions by the condition $\int_D P^2 = 1$.

Poincaré's clever proof has been rapidly forgotten, because of the development, by Ivar Fredholm et David Hilbert, of the *spectral theory of linear integral equations,* that it had strongly inspired. Informations about the developments of the spectral theory in the twentieth century can be found in [7], and more details on Poincaré's contributions are given in [16] and [40].

The next chapter of the memoir [29] studies the *maximum principle* for problem

$$\Delta v = -f \quad \text{in } D, \quad \frac{\partial v}{\partial n} + hv = \varphi \quad \text{on } S, \tag{15}$$

when $f > 0$, $h > 0$ and $\varphi > 0$. Its most original contribution is the introduction of the idea of *weak solution* of problem (15). Poincaré observed that if v is a solution of (15) and u is an arbitrary function twice continuously differentiable, then the relation

$$\int_D uf\,dx + \int_D v\Delta u\,dx + \int_S v\varphi\,d\sigma = \int_S v\left(hu + \frac{\partial u}{\partial n}\right)d\sigma, \tag{16}$$

that Poincaré called the *modified condition,* holds. Conversely, if condition (16) holds for any u, the equations (15) are also satisfied, provided v and $\frac{\partial v}{\partial n}$ are continuous. Poincaré then showed that if v is just continuous and satisfies (16) for all u, the maximum principles developed in the classical frame remain valid. After having recalled the idea of Neumann's method for the Dirichlet problem on which we will come back later, Poincaré applied it to the mixed problem

$$\Delta v + f = 0 \quad \text{in } D, \quad \frac{\partial v}{\partial n} + hv = 0 \quad \text{on } S,$$

when D is a bounded convex open set. He did not succeed in proving that the solution v obtained by this method of successive approximations has a normal derivative on S and, *a fortiori*, that it satisfies the boundary condition. But he proved that it satisfies the condition (16) (with $\varphi = 0$), anticipating again the concept of weak solution when writing: "We are not sure that the expression $\frac{\partial v}{\partial n}$ has a meaning and that the boundary condition $\frac{\partial v}{\partial n} + hu = 0$ is fulfilled. But we can assert that we have the *modified condition*. It is obviously physically equivalent."

Finally, the remaining part of the memoir was a tentative extension of his proof for the existence of the eigenvalues to the problem

$$\Delta v + \xi v + f = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} + hv = 0 \quad \text{on } S,$$

which met the same difficulty concerning the regularity of the coefficients v_n on the boundary. Poincaré left the complete solution to his followers: "One can see that I have not been able to obtain in the general case results as satisfactory as in the case $h = \infty$; one can see how many gaps still remain. I will not try to fill them more [...]. I think however that those results, even incomplete, were not completely without interest, and I have decided to publish them. I would be happy if this publication could suggest new researches on this topics."

No doubt that Poincaré's desire has been fully realized. The memoir ends with some partial results on the expansion of a function in series of harmonic functions.

Neumann's method for the Dirichlet problem on an arbitrary domain

In 1878, Carl Neumann [20] had justified mathematically a formal method, introduced by Beer around 1860, to obtain the existence of a solution to the Dirichlet problem

$$\Delta V = 0 \quad \text{in } D, \quad V = \Phi \quad \text{on } S, \tag{17}$$

when Φ is continuous and D is bounded, convex, with sufficiently regular boundary S. The method consisted in searching V in the form of a *double layer potential* of unknown density ρ, i.e. as a surface integral

$$V(x) = \int_S \rho(s) \frac{\partial}{\partial n} \frac{1}{\|x-s\|}\,d\sigma.$$

It is well known that V is harmonic outside S, and that, for $y \in S$, the limits

$$V^i(y) = \lim_{x \to y; x \in D} V(x), \quad V^e(y) = \lim_{x \to y; x \notin \overline{D}} V(x)$$

exist and satisfy $V^i(y) - V^e(y) = -4\pi\rho(y)$. Letting, for $y \in S$, $V(y) = \frac{1}{2}[V^i(y) + V^e(y)]$, Neumann tried to determine V in such a way that, on S, the relation

$$V^i - V^e = \lambda(V^i + V^e) + 2\Phi \tag{18}$$

holds for some real parameter λ. If the problem is solved for $\lambda = -1$, then $V^i = \Phi$ on S, and V is a solution of (17). Searching V as a series $V = \sum_{n=0}^{\infty} \lambda^n V_n$, which gives $V^i = \sum_{n=0}^{\infty} \lambda^n V_n^i$, $V^e = \sum_{n=0}^{\infty} \lambda^n V_n^e$, and, on S, $V_n = \frac{1}{2}(V_n^i + V_n^e)$, one finds, identifying in (18) the coefficients of the similar powers of λ, the recurrence relations, for $x \in S$,

$$\begin{aligned} V_0(x) &= -\int_S \frac{\Phi(s)}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|}\,d\sigma, \\ V_n(x) &= -\int_S \frac{V_{n-1}(s)}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|}\,d\sigma \ (n = 1, 2, \ldots). \end{aligned} \tag{19}$$

When D is convex, Neumann proved the existence of a constant C such that the series $\sum_{n=0}^{\infty} \lambda^n (V_n - C)$ converges for $|\lambda| \le 1$. Poincaré's memoir [33] of 1896, announced in note [32] of 18 February 1895, proved the convergence of Neumann's method when D *is simply connected and* S *has at each point a tangent plane and two curvature radii.* We do not give the details of this highly technical and today almost forgotten proof. The historical importance of [33] is, together with [29], to have inspired Ivar Fredholm in building his theory of *linear integral equations*. Indeed, the relations (19) give

$$\sum_{n=1}^{\infty} \lambda^n V_n(x) = -\lambda \int_S \frac{1}{2\pi} \sum_{n=1}^{\infty} \lambda^{n-1} V_{n-1}(s) \frac{\partial}{\partial n} \frac{1}{\|x-s\|}\, d\sigma,$$

i.e.

$$V(x) + \lambda \int_S \frac{1}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|} V(s)\, d\sigma + \int_S \frac{1}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|} \Phi(s)\, d\sigma = 0.$$

In other terms, V satisfies what is now called a *Fredholm integral equation of second kind* with kernel $\frac{1}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|}$. Paper [9] where Fredholm introduced his first study of an integral equation of second kind, entitled 'Sur une nouvelle méthode pour la résolution du problème de Dirichlet', starts as follows:

"In his profound researches (*Acta Mathematica* t. 20) on the convergence of Neumann's well-known method in potential theory, Mr. Poincaré has considered the Dirichlet problem as a special case of another problem, that he calls Neumann's problem. [...] Neumann has solved this problem in expanding the unknown function according to the power of a parameter λ. But it follows from Mr. Poincaré's researches that V is a meromorphic function. Hence it is clear that Neumann's series expansion cannot converge for all values of λ. But because we know that a meromorphic function can always be written as a quotient of two entire functions, I have found natural to search directly those entire functions."

Poincaré immediately grasped the importance of Fredholm's work on integral equations, generalized it in several ways and applied it to the theory of tides and to the diffraction of Hertzian waves. Half of Poincaré's lectures delivered in Göttingen in April 1909, upon Hilbert's invitation, deal with Fredholm integral equations and their applications [38].

Heat equation and telegraph equation

In the third section 'The laws of cooling' of his memoir [26], Poincaré considered the *non-stationary Fourier problem*, namely finding $V = V(t,x)$ such that $V(0,x) = V_0(x)$ $(x \in D)$ and

$$\begin{aligned} \frac{\partial V}{\partial t} &= a^2 \Delta V \quad \text{in }]0,\infty[\times D, \\ \frac{\partial V}{\partial n} + hV &= 0 \quad \text{on }]0,\infty[\times S, \end{aligned} \tag{20}$$

If the U_n are the fundamental functions of the stationary problem and k_n the corresponding characteristic numbers $(n = 1,2,\ldots)$, one should be able to expand V_0 as $V_0 = \sum_{n=1}^{\infty} A_n U_n$, to deduce that, for all $t > 0$, $V(t,x) = \sum_{n=1}^{\infty} A_n e^{-a^2 k_n t} U_n(x)$. Observing that he "was not yet able to prove the possibility of the expansion in a general setting", Poincaré showed that the *averaged error*

$$S_0 := \int_D \left[V_0 - \sum_{p=0}^{n} A_p U_p \right]^2$$

is minimal when $A_p = \int_D V_0 U_p := J_p^0$ $(1 \le p \le n)$. For this choice of the coefficients A_p, if $J_p(t) = J_p^0 e^{-a^2 k_p t}$, Poincaré showed that one can take n large enough so that the averaged error made on the temperature at time t,

$$S(t) := \int_D \left[V(t,x) - \sum_{p=1}^{n} J_p(t) U_p(x) \right]^2 dx,$$

can be made as small as one wants. One recognizes in those considerations the beginning of the use of a *Hilbertian norm* instead of a uniform one in the study of a parabolic problem.

In an interesting note [27] in the *Comptes Rendus* of 26 December 1893, Poincaré gave the first general solution of the initial value problem for the *telegraph equation* on an indefinite line

$$\frac{\partial^2 V}{\partial t^2} + 2\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \tag{21}$$

which describes the propagation of an electrical current in a conducting wire. This equation had been introduced in 1857 by Gustav Kirchhoff [12], starting from Weber's electromagnetic theory, and deduced from Maxwell's theory by Oliver Heaviside [11] in 1876. Those authors had only given special solutions of this equation. It is noticeable that Poincaré's motivation is not wire telegraphy, but to give a theoretical explanation to some experimental discrepancies in measuring the speed of propagation of electricity in wires. Letting $V = Ue^{-t}$, equation (21) becomes

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + U. \tag{22}$$

Given the initial conditions $U(0,x) = f(x)$ and $\frac{\partial U}{\partial t}(0,x) = f_1(x)$ in the form of Fourier integrals

$$f(x) = \int_{-\infty}^{+\infty} \theta(q) e^{iqx}\, dq, \quad f_1(x) = \int_{-\infty}^{+\infty} \theta_1(q) e^{iqx}\, dq,$$

searching the solution $U(x,t)$ in the form of a Fourier integral, Poincaré classically arrived to the expression

$$U(x,t) = \int_{-\infty}^{+\infty} e^{iqx} \left[\theta(q) \cos(t\sqrt{q^2-1}) + \theta_1(q) \frac{\sin(t\sqrt{q^2-1})}{\sqrt{q^2-1}} \right] dq$$

or, in complex form,

$$U(x,t) = \int_{-\infty}^{+\infty} \alpha(q) e^{i[qx + t\sqrt{q^2-1}]}\, dq + \int_{-\infty}^{+\infty} \beta(q) e^{i[qx - t\sqrt{q^2-1}]}\, dq$$

with

$$\alpha(q) = \frac{\theta(q)}{2} + \frac{\theta_1(q)}{2i\sqrt{q^2-1}}, \quad \beta(q) = \frac{\theta(q)}{2} - \frac{\theta_1(q)}{2i\sqrt{q^2-1}}.$$

For initial conditions $\theta \equiv 0$ and $\theta_1 \equiv 1$, contour integration allowed him to find the expression

$$U(x,t) = \begin{cases} 0 & \text{if } x > t \\ \Lambda(x,t) & \text{if } -t < x < t \\ 0 & \text{if } x < -t \end{cases}$$

with

$$\Lambda(x,t) := \frac{1}{2}\int_0^{2\pi} e^{ix\cos\varphi} e^{i\sin\varphi}\, d\varphi = \pi J_0(\sqrt{x^2-t^2}),$$

and J_0 is the Bessel function of index 0.

Poincaré then obtained a formula for the solution in the case where $f \equiv 0$ and $f_1(x) \neq 0$ if $b \le x \le a$, $f_1(x) = 0$ if $x < b$ or if $x > a$, and in the case where $f_1 \equiv 0$ and $f(x) \neq 0$ if $b \le x \le a$, $f(x) = 0$ if $x < b$ or if $x > a$. The combination of those two solutions gives the general solution when f and f_1 have compact support, and a detailed analysis of this solution allowed him to describe precisely the behaviour of an initial perturbation bounded in time and space: "One first sees that the head of the perturbation propagates with some speed, in such a way that, before this head, the perturbation is zero, in contrast to what happens in Fourier's heat theory, and according to the laws of propagation of light or sound through plane waves, deduced from the vibrating string equation. But there is, with respect to this last case, an important difference, because the perturbation, during its propagation, leaves behind a residue which is not zero [...] and can then trouble the observations."

So it is the diffusion of the wave in the wire which makes difficult the measure of its speed.

The study of the telegraph equation was considered with more details in [30–31, 35]. See [17] for a more complete description of Poincaré's contributions to the telegraph equation, and of his motivations.

A non-linear elliptic equation

The theory of Fuchsian functions leads to the study of solutions of the partial differential equation $\Delta u = ke^u$ $(k > 0)$, already considered by Joseph Liouville between 1847 and 1853, and by Émile Picard between 1890 and 1893. It is related to the theory of surfaces with constant negative curvature, closely linked to Lobatchevsky's geometry, and therefore to Fuchsian functions. Although Poincaré, in a memoir [37] of 1898, announced in a note [36] of 28 February of the same year, considered this equation on a Klein surface, we present his ideas, for simplicity, in the analogous more standard case of the homogeneous Neumann problem on a bounded planar domain D with boundary S.

Let θ be a positive smooth function and Φ a regular function on $\overline{D} := D \cup S$. The problem consists in finding a function u such that

$$\begin{aligned} &\Delta u = \theta(x)e^u - \Phi(x) \quad \text{in } D, \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } S. \end{aligned} \tag{23}$$

Starting from the fact that Neumann's problem for equation

$$\Delta u = \varphi(x) \quad \text{in } D \tag{24}$$

is solvable if and only if $\int_D \varphi = 0$, Poincaré first considered Neumann's problem for the *linear equation*

$$\Delta u = \eta u - \varphi(x) \quad \text{in } D, \tag{25}$$

where $\eta > 0$. To do so, he introduced the family of equations depending upon the parameter $\lambda > 0$,

$$\Delta u = \lambda\eta u - \varphi(x) - \lambda\psi(x) \quad \text{in } D, \tag{26}$$

with $\int_D \varphi = 0$, and showed that a unique solution of Neumann's problem for (26), of the form $u = \sum_{k=0}^{\infty} u_k \lambda^k$, exists if $2\beta\lambda < 1$, with $\beta > 0$ a constant related to problem (24). Assuming then that Neumann's problem for equation

$$\Delta u = \eta u - \Phi(x) \quad \text{in } D$$

has a unique solution for every Φ, Poincaré showed that the same problem for

$$\Delta u = (\eta + \lambda\eta')u - \Phi(x) \quad \text{in } D \tag{27}$$

with $\eta' > 0$ has a unique solution, of the form $u = \sum_{k=0}^{\infty} u_k \lambda^k$, when $\lambda < \eta/\eta'$.

To deal with (25), Poincaré introduced p positive numbers $\lambda_1, \ldots, \lambda_p$ such that $\sum_{k=1}^{p} \lambda_k = 1$, and wrote $\varphi(x) = \varphi_1(x) + \lambda_1\psi_1(x)$, with $\int_D \varphi_1 = 0$. From the result on (26), Neumann's problem for equation

$$\Delta u = \lambda_1\eta u - \varphi_1(x) - \lambda_1\psi_1(x) \quad \text{in } D$$

has a unique solution if $\lambda_1 < 1/2\beta$. From the result on (27), the same is true for equation

$$\Delta u = \lambda_1\eta u + \lambda_2\eta u - \varphi(x) \quad \text{in } D$$

when $\lambda_2 < \lambda_1$, and for equation

$$\Delta u = (\lambda_1 + \lambda_2)\eta u + \lambda_3\eta u - \varphi(x) \quad \text{in } D$$

when $\lambda_3 < \lambda_1 + \lambda_2$. Continuing in the same way, Neumann's problem for equation

$$\Delta u = (\lambda_1 + \cdots + \lambda_{p-1})\eta u + \lambda_p\eta u - \varphi(x) \quad \text{in } D,$$

i.e. for equation (25), has a unique solution if $\lambda_p < \lambda_1 + \cdots + \lambda_{p-1}$. The conditions on the λ_k can all be fulfilled by taking p sufficiently large. The underlying idea to this proof is clearly a *continuation method*.

Returning to the *non-linear problem* (23), integrating both members of the equation over D, using the Green formula and the boundary condition, Poincaré showed that $\int_D \Phi > 0$ was a *necessary* condition for the existence of a solution to (23). If U denotes a solution of Neumann's problem for the linear equation

$$\Delta u = -\Phi(x) + \frac{1}{|D|}\int_D \Phi \quad \text{in } D,$$

the change of variable $u = U + v$ leads to the equivalent Neumann problem for equation

$$\Delta v = \theta(x)e^{U(x)}e^{v} - \frac{1}{|D|}\int_D \Phi \quad \text{in } D,$$

which, together with the necessary condition, shows that it suffices to consider the case of a positive Φ.

Poincaré noticed that a first idea to prove the existence of a solution consists in using *Dirichlet's principle*, i.e. finding a function u minimizing the energy integral

$$\int_D \left[\frac{\|\nabla u(x)\|^2}{2} + \theta e^{u(x)} - \Phi(x)u(x)\right] dx,$$

bounded from below when Φ is positive, but the existence of a minimum is not guaranteed. Hence Poincaré returned to the *continuation method* already used for (25). Assuming that problem

$$\Delta u = \theta(x)e^{u} - \varphi(x) \quad \text{in } D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } S$$

has a unique solution u_0, Poincaré introduced the family of equations

$$\Delta u = \theta(x)e^{u} - \varphi(x) - \lambda\psi(x) \quad \text{in} \quad D, \tag{28}$$

and searched a solution for Neumann's problem in the form $u = \sum_{k=0}^{\infty} u_k\lambda^k$. Letting

$$e^u = 1 + u + \sum_{k=2}^{\infty} w_k\lambda^k,$$

he showed that $w_k = w_k(u_1, \ldots, u_{k-1})$ and that $\max_{T\cup S}|u_k| \le \max_{\overline{D}}|w_k|$ for every $k \ge 2$. The convergence of $\sum_{k=1}^{\infty} u_k\lambda^k$, follows, as well as the existence of a unique solution to (28) if

$$|\lambda| < (\log 4 - 1)\frac{\max_{\overline{D}}(|\varphi|/\theta)}{\max_{\overline{D}}(|\psi|/\theta)}.$$

But, for any $\alpha > 0$, the problem

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) \quad \text{in } D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } S$$

has the unique solution $u(x) = \log\alpha$. Hence, if ψ is positive, Neumann's problem for equation

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) - \lambda\psi(x) \quad \text{in } D$$

has a unique solution for $0 \le \lambda < \alpha(\log 4 - 1)/\max_{\overline{D}}(\psi/\theta)$, the same is true for

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) - \lambda\psi(x) - \mu\psi(x) \quad \text{in } D,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } S$$

if

$$0 \le \mu < (\log 4 - 1)\frac{\max_{\overline{D}}[\alpha + (\lambda\psi/\theta)]}{\max_{\overline{D}}(\psi/\theta)},$$

and hence in particular if $0 \le \mu < \lambda < \alpha(\log 4 - 1)/\max_{\overline{D}}(\psi/\theta)$. Continuing in this way, problem (28) has a unique solution for any $0 \le \lambda < n\alpha(\log 4 - 1)/\max_{\overline{D}}(\psi/\theta)$, and any $n \ge 1$. Taking now $\psi = \Phi - \alpha\theta$, with $0 < \alpha < \min_{D\cup S}(\Phi/\theta)$ in order that ψ be positive, and taking n such that $n\alpha(\log 4 - 1)/\max_{\overline{D}}[(\Phi - \alpha\theta)/\theta] > 1$, Neumann's problem for equation

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) - \lambda[\Phi(x) - \alpha\theta(x)] \quad \text{in } D$$

has a unique solution for $\lambda = 1$, which is the wanted result.

Notice that a very similar approach had been used and very clearly presented, for a Dirichlet problem associated to semi-linear elliptic equations, in the PhD thesis of Édouard Le Roy [13–14] (dedicated to Poincaré, who also wrote the report). Despite of the proximity of publication dates, neither author quoted the other one. As mentioned in the Introduction of [37], Poincaré had previously, like Félix Klein, used a continuation method, in a finite dimensional setting, in the theory of Fuchsian functions. The method and results of Le Roy and of Poincaré are briefly described by Arnold Sommerfeld in [42]. His short description starts with: "Finally, to deal with the equations $\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial z} = fu$ and $\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial z} = F(x, y, z, u)$, Ed. Le Roy has introduced the following method of analytic continuation..." and ends as follows: "H. Poincaré also uses the same ideas in his researches referred in Nr. 12 [$\Delta u = ke^u$]." Consequently, the introduction of a continuation method to study semi-linear elliptic boundary value problems, generally attributed to Sergei N. Bernstein's paper [4] of 1906, must be traced to Le Roy and Poincaré's contributions of 1898, which are not quoted in [4], despite of the very close relations of Bernstein with the French mathematical community. For more details on Le Roy's and Poincaré's contributions, and the development of the continuation method, see [18].

Partial differential equations in Poincaré's monographs

When he occupied the chair of mathematical physics and probabilities at the University of Paris, Poincaré changed almost every semester the topics of his lectures. Most of them were then written down by one or two (bright) students and revised by Poincaré before publication. Although partial differential equations are present in other ones, we only describe the most significant ones.

The telegraph equation is discussed in the monograph [30], describing Poincaré's lectures on electrical oscillations during the first semester of the academic year 1892–1893. It is edited by Charles Maurain (1871–1967), a future well-known geophysicist. The telegraph equation is introduced and discussed on pp. 182–189 of a 'Complement to Chapter IV' starting as follows: "Since this course has been given, several experiments have been done and I will be forced to add from time to time several supplementary lines to inform the reader about the present state of science; in this way, I must mention the nice experiments of M. Blondlot on the propagation of electricity."

After describing the experiments of Hippolyte Fizeau (1819–1896) and Eugène Gounelle (1821–1864) of 1850 in order to measure the speed of propagation of electricity in metallic wires, Poincaré mentioned that: "Everything happens as if the perturbation shaded off in such a way to occupy more space on the wire at the arrival than at the departure. This phenomenon, placed beyond any doubt by the experiments of M. Fizeau, has been called by this physicist the diffusion of the current. [...] It is likely that this diffusion is due to the Ohmic resistance that we have neglected up to now.”

Poincaré then showed how the equations of propagation of electricity are modified by the presence of this resistance, and arrived, for the variations of the electrical potential in a wire transmitting an electrical perturbation, to the equation

$$A\frac{\partial^2 V}{\partial t^2} + 2B\frac{\partial V}{\partial t} = C\frac{\partial^2 V}{\partial x^2},$$

which is known under the name of *telegraph equation* (*équation des télégraphistes* in French).

Poincaré then solved this equation in a way entirely similar to the one given in [27].

During the first semester of the academic year 1893–1894, Poincaré devoted his lectures to the analytical theory of the propagation of heat. They were written down by Rouyer and René Baire (1874–1932). The book starts as follows:

“Fourier’s heat theory is one of the first examples of the application of analysis to physics. [...] The results he has obtained are surely interesting by themselves, but what is still more interesting is the method he has used to reach them, which always will remain a model for those who will want to cultivate a branch of mathematical physics. I will add that Fourier’s book has a fundamental importance in the history of mathematics, and that pure analysis maybe owes him still more than applied analysis.”

After discussing the various types of propagation of heat, Poincaré established Fourier’s heat equation and gave it in various systems of coordinates. The case of an infinite rectangular solid was then treated by separation of variables and Fourier series (giving Poincaré an opportunity to recall Dirichlet’s and Abel’s contributions). Fourier series are then used to solve Fourier’s equation on a closed wire. The case of an infinite wire led Poincaré to Fourier’s integral whose main properties are established. Follows then an interesting chapter on ‘linear equations analogous to that of heat’, where the vibrating string and telegraph equations are solved using the method of Fourier integrals, and their solutions compared to those of heat equation in the case of an initial perturbation with compact support.

The heat equation on an infinite wire or solid is then solved by Laplace method, and the justification of Fourier’s formal results on the cooling of a sphere, led Poincaré to a description of Cauchy’s theory on asymptotic values of some integrals. The case of the cooling of an arbitrary body D with boundary S is reduced, by separation of time and space variables, to the problem of finding a non-trivial function $U(x,y,z)$ and a number k such that

$$\Delta U + kU = 0 \quad \text{in } D, \quad \frac{\partial U}{\partial n} + hU = 0 \quad \text{on } S.$$

For this problem, “losing in rigour what we will gain in generality”, Poincaré reproduced his results of the memoir [26] giving the corresponding eigenvalues and eigenfunctions by an extension of Dirichlet’s principle, and only quoted the second memoir [29] where the rigorous proof of existence of eigenvalues and eigenfunctions is given. The remaining part of the book is devoted to the use of special functions in solving various problems of analytical heat theory.

Poincaré’s lectures of the first semester of the academic year 1894–1895, devoted to the theory of Newtonian potential, were written down by Édouard Le Roy and Georges Vincent. The first part of the book gives a rather standard presentation of the computation of the various integrals expressing the Newtonian potential for volumes, surfaces and lines. As a curiosity (called to my attention by Michel Willem), one should mention Poincaré’s remark that many of the used results on particular integrals of potential theory are a special case of the following theorem on integrals, stated on p. 121:

Theorem. *Consider the integral*

$$\int_{(S)} f(x,y,z)\,dx\,dy$$

extended to some domain S. Assume that the following conditions hold:

1. *The curve C bounding S does not depend upon z.*
2. *One has, at any point of S,*

$$f(x,y,z) < \varphi(x,y)$$

where φ is a positive function.

3. *The integral $\int \varphi(x,y)\,dx\,dy$, extended to the domain S, exists.*
4. *Finally one has*

$$\lim_{z\to 0} f(x,y,z) = f(x,y,0)$$

for any fixed x, y.

In those conditions, one has the relation

$$\lim_{z=0} \int_{(S)} f(x,y,z)\,dx\,dy = \int_{(S)} f(x,y,0)\,dx\,dy.$$

The reader has already noticed the similarity of the statement with that of Lebesgue’s dominated convergence theorem, even if the background here is, of course, not Lebesgue integration theory, but some improper integrals. Poincaré then described the properties of harmonic functions (mean value and maximum principles), and the use of Green’s functions in solving the Dirichlet problem, followed by the special case of the circle and the sphere, and the double layer potential.

The last hundred pages of the book are more closely related to Poincaré’s recent researches in potential theory, giving a presentation of his results of [26] on the sweeping out method (Chapter VII), Neumann’s method on convex sets (Chapter VIII), and his extension of Neumann’s method to simply connected domains given in [33] (Chapter IX).

This short description suffices to show that, in his lectures, Poincaré not only gave a masterly presentation of the classical theory of partial differential equations of mathematical physics and their evolution, but did not hesitate to present his most recent results in this area. It is hard to find a better illustration of the necessary and wonderful interaction between research and teaching at the university level.

Conclusion

Poincaré’s contributions to partial differential equations would have been amply sufficient to place him among the greatest mathematicians of the end of the nineteenth century and the beginning of the twentieth century. But Poincaré is also the father of automorphic functions, dynamical systems and chaos, algebraic topology and modern celestial mechanics, to quote only his main achievements [15]. He is one of the greatest mathematicians of history, and, in addition, his important contributions to physics and to the philosophy of science place him at a high level in those disciplines.

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Self-excited oscillations: from Poincaré to Andronov

In 1908 Henri Poincaré gave a series of 'forgotten lectures' on wireless telegraphy in which he demonstrated the existence of a stable limit cycle **in the phase plane. In 1929 Aleksandr Andronov published a short note in the** Comptes Rendus **in which he stated that there is a correspondence between the periodic solution of self-oscillating systems and the concept of stable limit cycles introduced by Poincaré. In this article Jean-Marc Ginoux describes these two major contributions to the development of non-linear oscillation theory and their reception in France.**

Until recently, the young Russian mathematician Aleksandr Andronov was considered by many scientists as the first to have applied the concept of *limit cycle*, introduced almost half a century before by Henri Poincaré, in order to state the existence of self-sustained oscillations.

Consequently, if the discovery of a series of 'forgotten lectures' given by Poincaré at the École Supérieure des Postes et Télégraphes (today Télécom ParisTech) in 1908 proves that he had applied his own concept of *limit cycle* to a problem of wireless telegraphy twenty years before Andronov, it reopens the discussion of Poincaré's French legacy in dynamical system theory or, more precisely in non-linear oscillations theory.

Poincaré's 'forgotten lectures' will be presented in the next section. The subsequent section goes into their reception in France before World War I and in the 1920's in the French engineers community. Special attention will be paid to the role of Jean-Baptiste Pomey who asked Élie Cartan to solve a problem of sustained oscillations in an electronic oscillator and, to the particular case of Alfred Liénard who proved, under certain conditions, existence and uniqueness of a periodic solution for such an oscillator without having regarded this periodic solution as a 'Poincaré's limit cycle'.

Starting from 1929, Andronov's note at the *Comptes Rendus* seems to have become the reference in terms of connection with Poincaré's works. So, the reception of Andronov's results by the French scientific community will be analysed in the before last section in considering a selection of works published in France between 1929 to 1943.

In the last section, the fact that Poincaré's lectures have been forgotten as well as the fact that neither Cartan nor Liénard have made any connection with Poincaré's works although they have obviously used some of them, will be discussed. Moreover, the role of Jacques Hadamard in the diffusion in France of the works of the Russian schools and of Poincaré's methods will be also pointed out. Thus, the question of Poincaré's legacy will appear in a new perspective.

Poincaré's forgotten lectures

On 4 July 1902 Poincaré became Professor of Theoretical Electricity at the École Supérieure des Postes et Télégraphes in Paris where he taught until 1910. The director of this school, Édouard Éstaunié (1862–1942), also asked him to give a series of conferences every two years in May–June from 1904 to 1912. He told about Poincaré's first lecture of 1904:

"Dès les premiers mots, il apparut que nous allions assister au travail de recherche de cet extraordinaire et génial mathématicien ... À chaque obstacle rencontré, une courte

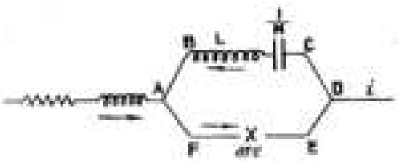

Figure 1 Circuit diagram of the singing arc, Poincaré [31, p. 390]

pause marquait l'embarras, puis d'un coup d'épaule, Poincaré semblait défier la fonction gênante ..."

In 1908 Poincaré chose as subject: wireless telegraphy. The text of his lectures was first published weekly in the journal *La Lumière Électrique* [31] before being edited as a book a year later [32]. In the fifth and last part of these lectures entitled 'Télégraphie dirigée: oscillations entretenues' (Directive telegraphy: sustained oscillations) Poincaré stated a necessary condition for the establishment of a stable regime of sustained oscillations in the singing arc (a forerunner device of the triode used in wireless telegraphy). More precisely, he demonstrated the existence, in the phase plane, of a *stable limit cycle*.

The singing arc equation

Starting from the following diagram (see Figure 1), Poincaré [31, p. 390] explained that this circuit consists of an Electro Motive Force (E.M.F.) of direct current E, a resistance R and a self-induction, and in parallel, a singing arc and another self-induction L and a capacitor. In order to provide the differential equation modeling the sustained oscillations he calls x the capacitor charge and i the current in the external circuit.

Thus, the current in the branch (ABCD) including the capacitor of capacity $1/H$ may be written: $x' = dx/dt$. The current intensity i_a in the branch (AFED) including the singing may be written while using Kirchhoff's law: $i_a = i + x'$. Then, Poincaré established the following second order non-linear differential equation for the sustained oscillations in the singing arc:

$$Lx'' + \rho x' + \theta(x') + Hx = 0. \qquad (1)$$

He specified that the term $\rho x'$ corresponds to the internal resistance of the self and various damping while the term $\theta(x')$ represents the E.M.F. of the arc which is related to the intensity by a function, unknown at that time.

Stability condition

Poincaré established, twenty years before Andronov [3], that the stability of the periodic solution of the above equation depends on the existence of a closed curve, i.e. of a *stable limit cycle* in the phase plane he has defined in his memoirs 'Sur les Courbes définies par une équation différentielle' [29, p. 168]. He posed:

$$x' = \frac{dx}{dt} = y, dt = \frac{dx}{y}, x'' = \frac{dy}{dt} = \frac{ydy}{dx}.$$

Thus, equation (1) becomes:

$$Ly\frac{dy}{dx} + \rho y + \theta(y) + Hx = 0. \qquad (2)$$

Poincaré [31, p. 390] stated that: "Sustained oscillations correspond to closed curves if there exist any." He gave the representation shown in Figure 2 for the solution of equation (2).

Let's notice that this closed curve is only a *metaphor* of the solution since Poincaré does not use any graphical integration method such as *isoclines*.

Poincaré explained that if $y = 0$, then dy/dx is infinite and so, the curve admits vertical tangents. Moreover, if x decreases, x' i.e. y is negative. He concluded that the trajectory curves turns in the direction indicated by the arrow (see Figure 2) and wrote:

"*Stability condition.* Let's consider another non-closed curve satisfying the differential equation, it will be a kind of spiral curve approaching indefinitely near the closed curve. If the closed curve represents a stable regime, by following the spiral in the direction of the arrow one should be brought back to the closed curve, and provided that this condition is fulfilled the closed curve will represent a stable regime of sustained waves and will give rise to a solution of this problem."

It clearly appears that the closed curve which represents a stable regime of sustained oscillations is nothing else but a limit cycle as Poincaré [26, p. 261] has introduced in his own famous memoir 'On the curves defined by differential equations' and as Poincaré [27, p. 25] has later defined in the notice on his own scientific works.

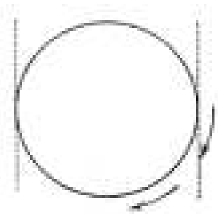

Figure 2 Closed curve solution of equation (2), Poincaré [31, p. 390]

But this first *giant step* is not sufficient to prove the stability of the oscillating regime. Poincaré had to demonstrate now that the periodic solution of equation (1) (the closed curve) corresponds to a *stable limit cycle*.

Possibility condition

In the following part of his lectures, Poincaré gave what he calls a 'condition de possibilité du problème'. In fact, he established a stability condition of the periodic solution of equation (1), i.e. a stability condition of the limit cycle under the form of inequality. After multiplying equation (2) by $x'dt$ Poincaré integrated it over one period while taking into account that the first and fourth term vanish since they correspond to the conservative part of this non-linear equation. He obtained:

$$\rho \int x'^2 dt + \int \theta(x')\, x' dt = 0.$$

He explained that since the first term is quadratic, the second one must be negative in order to satisfy this equality. So, he stated that the oscillating regime is stable iff

$$\int \theta(x')\, x' dt < 0. \qquad (3)$$

As exemplified below, Poincaré's approach is identical to the one which will be used by Alfred Liénard twenty years later.

Reception of Poincaré's lectures in France

The discovery of these Poincaré's 'forgotten lectures' on wireless telegraphy implied the analysis of their influence on the French engineers community.

Before World War I (1910–1914)

During this period one scientific reference to these conferences could be found. It is in the book by Gaston Émile Petit and Léon Bouthillon entitled *La Télégraphie Sans Fil*, published in 1910, in which one can read [24, p. 128]:

"Le problème de la direction des ondes est donc résoluble. Les ondes peuvent théoriquement être concentrées en faisceau comme les rayons lumineux par des dispositifs appropriés ([1]).

([1]) Poincaré, Conférences sur la Télégraphie sans fil faites à l'école professionnelle supérieure des Postes et Télégraphes de Paris, 1908, p. 25."

Unfortunately, this unique quotation is very disappointing since it does not refer to the part of Poincaré's lectures concerning the sustained oscillations. Nevertheless, since Petit (ESPT, 1906) and Bouthillon (ESPT, 1907) were students at the École Supérieure des

Postes et Télégraphes (ESPT) when Poincaré was teaching there, one can suppose that they could have attended his lectures of 1908.

There are several allusions to his lectures in the eulogies made at the time of Poincaré's death on 17 July 1912 and after.

This is the case for example of the French engineer André Blondel (1863–1938) who wrote on 27 July a tribute to Poincaré [6, p. 100]:

"De même aussi il avait été conduit à étudier dans ses *Conférences à l'École Supérieure des Postes et Télégraphes* le problème de la propagation de l'électricité, à propos duquel il a développé les recherches de Kohlrausch et poussé plus loin ses résultats. De même il fut amené à s'intéressé à la télégraphie sans fil, qui était pour lui une application des théories qu'il avait développées sur les oscillations électriques."

The following year, Gaston Darboux (1842–1917) in his historical praise recalled [8, p. 37]:

"Les conférences qu'il a données à l'École de Télégraphie nous montrent également combien il se tenait près de l'expérience, et quels services il a rendus aux praticiens.

L'équation, dite des télégraphistes, nous fait connaître, comme on sait, les lois de la propagation d'une perturbation électrique dans un fil. Poincaré intègre cette équation par une méthode générale qui peut s'appliquer à un grand nombre de questions analogues. Le résultat varie suivant la nature du récepteur placé sur la ligne, ce qui se traduit mathématiquement par un changement dans les équations aux limites, mais la même méthode permet de traiter tous les cas.

Dans une seconde série de conférences, Poincaré a étudié le récepteur téléphonique; un point qu'il a mis particulièrement en évidence, c'est le rôle des courants de Foucault dans la masse de l'aimant.

Enfin, dans une troisième série de conférences, il a traité les diverses questions mathématiques relatives à la télégraphie sans fil: émission, champ en un point éloigné ou rapproché, diffraction, réception, résonance, ondes dirigées, *ondes entretenues* ([1]).

([1]) Ces Conférences ont été publiées dans la collection des cours de l'école et dans la revue *L'Éclairage électrique*."

After World War I (1919–1928)
One might think that his lectures were forgotten because they dealt with an application to a device for wireless telegraphy: the singing arc which had become obsolete after the war.

Janet and the electrical-mechanical analogy (1919). Of course, during World War I the triode invented on 15 January 1907 by Lee de Forest (1873–1961) had supplanted the singing arc. However, a French engineer named Paul Janet (1863–1937) published in 1919 a note at the *Comptes Rendus* in which he established an analogy between the triode and the singing arc. In this paper, entitled 'Sur une analogie électrotechnique des oscillations entretenues' and, by using the classical electrical-mechanical analogy, Janet [13] provided the non-linear differential equation characterizing the oscillations sustained by the singing arc and by the triode:

$$L\frac{d^2i}{dt^2}+[R-f'(i)]\frac{di}{dt}+\frac{k^2}{K}i=0. \quad (4)$$

By using the electrical-mechanical analogy and by posing $H \to k^2/K$ and $\theta(x')=f'(i)\frac{di}{dt}$ it is easy to show that equation (1) and equation (4) are completely analogous. Nevertheless, there is no reference to Poincaré in this note.

Pomey and the singing arc equation (1920). On 28 June 1920, a textbook written by the engineer Jean-Baptiste Pomey (1861–1943) entitled 'Introduction à la théorie des courants téléphoniques et de la radiotélégraphie' was published in France. A former student of the École Supérieure des Postes et Télégraphes, Pomey (ESPT, 1883) became Professor of Theoretical Electricity in this school alongside Henri Poincaré and then director from 1924 to 1926. In Chapter XIX of his book, devoted to the generation of sustained oscillations Pomey [35, p. 375] wrote:

"Pour que des oscillations soient engendrées spontanément et s'entretiennent, il ne suffit pas que l'on ait un mouvement périodique, il faut encore que ce mouvement soit stable."

He provided the non-linear differential equation of the singing arc:

$$Lx''+Rx'+\frac{1}{C}x=E_0+ax'-bx'^3. \quad (5)$$

By posing $H=1/C$, $\rho=R$ and $\theta(x')=-E_0-ax'+bx'^3$, it is obvious that equation (1) and equation (5) are completely identical (for more details see [11]). Moreover, it is striking to observe that Pomey has used exactly the same variable x' as Poincaré to represent the current intensity. Here again, there is no reference to Poincaré. This is very surprising since Pomey was present during the last lecture of Poincaré at the École Supérieure des Postes et Télégraphes in 1912 for which he had written the

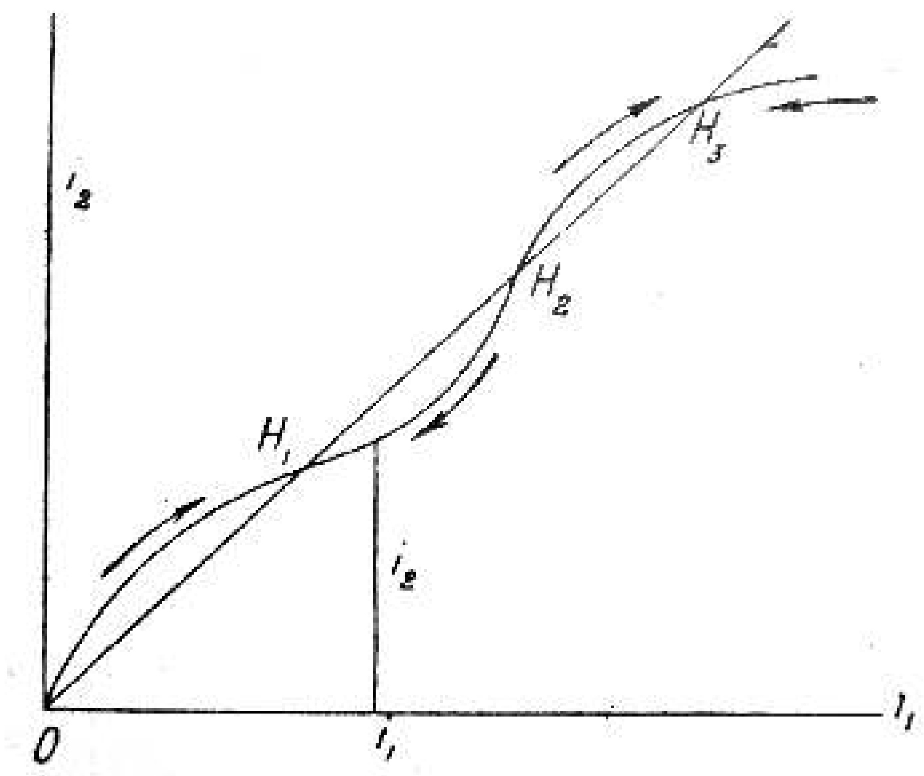

Figure 3 'First return map' diagram, [7, p. 1199]

Figure 4 Closed curve solution of equation (6), Liénard [17, p. 905].

introduction. So, one can imagine that he could have attended the lecture of 1908.

Élie and Henri Cartan and the existence of a periodic solution (1925). On 28 September 1925, Pomey wrote a letter to the mathematician Élie Cartan (1869–1951) in which he asked him to provide a condition for which the oscillations of an electronic device analogous to the singing arc and to the triode whose equation is exactly that of Janet (see equation (4)) are sustained. Within ten days, Élie Cartan and his son Henri sent an article entitled: 'Note sur la génération des oscillations entretenues' in which they proved the existence of a periodic solution for Janet's equation (4). In fact, their proof is based on a diagram (see Figure 3) which corresponds exactly to a 'first return map' diagram introduced by Poincaré in his memoir 'Sur les Courbes définies par une équation différentielle' [26, p. 251]. They wrote [7, p. 1199]:

"Les points H_1, H_2, H_3 où la courbe coupe la bissectrice correspondent à des solutions *périodiques* (oscillations entretenues) *dont l'existence est ainsi démontrée*. Elles sont périodiques parce qu'en partant d'un minimum donné $-i_1$, le maximum suivant est égal à i_1, par suite le minimum suivant e à $-i_1$, etc. On peut maintenant voir facilement que *toute solution tend vers une solution périodique*."

Obviously, Figure 3 exhibits an application of Poincaré's method although there is no reference to his works. Moreover, Cartan didn't recognize in this periodic solution a limit cycle of Poincaré.

Liénard and the existence and uniqueness of a periodic solution (1928). Three years later, on May 1928, the engineer Alfred Liénard (1869–1958) published an article entitled 'Étude des oscillations entretenues' in which he studied the solution of the following non-linear differential equation:

$$\frac{d^2x}{dt^2} + \omega f(x) \frac{dx}{dt} + \omega^2 x = 0. \qquad (6)$$

Such an equation is a generalization of the well-known Van der Pol's equation and of course of Janet's equation (4). Under certain assumptions on the function $F(x) = \int_0^x f(x)\,dx$ less restrictive than those chosen by Cartan [7] and Van der Pol [33], Liénard [17] proved the existence and uniqueness of a periodic solution of equation (6). Liénard [17, p. 906] plotted this solution (see Figure 4) and wrote:

"On se rend ainsi compte que la courbe intégrale décrit une sorte de spirale tendant asymptotiquement vers la courbe fermée D. Pour les courbes intégrales extérieures à la courbe fermée, c'est OA_2 qui devient inférieur à OA_1. La courbe se rapproche encore de la courbe D, mais par l'extérieur. En raison de ce fait que toutes les courbes intégrales, intérieures ou extérieures, parcourues dans le sens des temps croissants, tendent asymptotiquement vers la courbe D on dit que le mouvement périodique correspondant est un mouvement *stable*."

Liénard [17, p. 906] explained that the condition for which the 'periodic motion' is stable is given by the following inequality:

$$\int_\Gamma F(x)\,dy > 0. \qquad (7)$$

By considering that the trajectory curve describes the closed curve clockwise in the case of Poincaré and counter clockwise in the case of Liénard, it is easy to show that both conditions (3) and (7) are completely identical (for more details see [10] and [12]) and represent an analogue of what is now called 'orbital stability'. Again, one can find no reference to Poincaré in Liénard's paper. Moreover, it is very surprising to observe that he didn't use the terminology 'limit cycle' to describe its periodic solution. The case of Liénard is really a riddle that we will discussed below.

Reception of Andronov's results in France

After Poincaré's death in 1912, the French mathematician Jacques Hadamard (1865–1963) succeeded him at the Academy of Sciences. According to Maz'ya et Shaposhnikova [20, p. 181] Hadamard was elected corresponding member of the Russian Academy of Sciences in 1922 and, foreign member of the Academy of Sciences of the USSR in 1929. That is probably the reason why he was asked to review (the original of this note is presented in [10, p. 494-496]) and to present at the *Comptes Rendus* on 14 October 1929 a note from the young Russian mathematician Aleksandr Aleksandrovich Andronov (1901–1952) entitled 'Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues' [3].

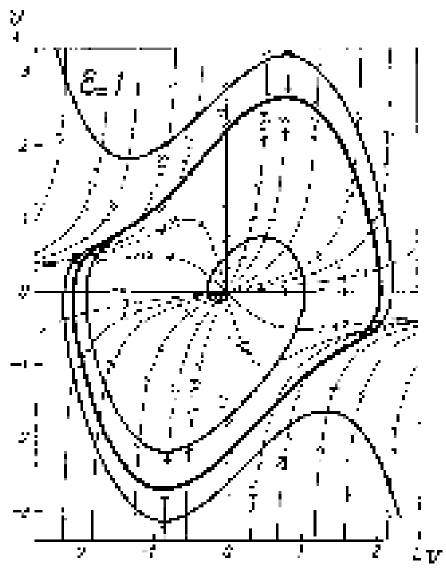

Figure 5 Graphical integration of the triode oscillator, Van der Pol [33, p. 983]

It has been pointed out by Ginoux [10, p. 176] that this note has been preceded by a presentation at the Congress of Russian Physicists between 5 and 16 August 1928 (see [2]).

In this short paper (only three pages), Andronov [3] stated a correspondence between the periodic solution of self-oscillating systems representing for example the oscillations of a radiophysics device used in wireless telegraphy and the concept of stable *limit cycles* introduced by Poincaré [26, p. 261].

In order to analyse the reception of Andronov's result, a selection of works published in France and elsewhere during the period (1930–1943) will be briefly presented.

Van der Pol's relaxation oscillations and limit cycles (1930). Looking at the famous plot of the graphical integration of the triode oscillator exhibited by Van der Pol [33, p. 983] (see Figure 5), one might think that he has immediately recognized that such a periodic solution was a *stable limit cycle* of Poincaré. This is not the case. It is just after the publication of Andronov's note at the *Comptes Rendus* that Van der Pol realized that the periodic solution he has plotted (see Figure 5) was a limit cycle. It was during a lecture given in Paris at the École Supérieure d'Électricité on 10 and 11 March 1930. Van der Pol [34, p. 294] reproduced the figures of his original paper of 1926 (including Figure 5) and he said:

"On remarque sur chacune de ces trois figures une courbe intégrale fermée; c'est un exemple de ce que Henri Poincaré a appelé un *cycle limite* (1), parce que les courbes intégrales s'en rapprochent asymptotiquement.

(1) Voir par exemple : *A. Andronow*, Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues, *Comptes Rendus* 189, p. 559 (1929).”

Let’s notice that although he quotes Poincaré, he makes reference to Andronov. Moreover, he also quotes in the following, the papers of Cartan [7] and that of Liénard [17] but it does not seem that he has ever used their works. During his visits in Paris, Van der Pol was hosted by a young French engineer named Philippe Le Corbeiller (1891–1980) who helped him to translate his talks (see Van der Pol [34, p. 312]). Le Corbeiller, who was probably present at the École Supérieure d’Électricité on 10 and 11 March 1930, was invited to give a lecture at the *Third International Congress of Applied Mechanics* held in Stockholm from 24 to 29 August 1930. He was accompanied by Van der Pol himself and by Alfred Liénard.

Le Corbeiller and the Theory of Non-linear Oscillations (1930). During his talk entitled ‘Sur les oscillations des régulateurs’ Le Corbeiller [14, p. 211] recalled:

“Si nous savons que le système machine-régulateur présente effectivement des oscillations périodiques, cela signifiera que parmi les courbes intégrales tracées sur la surface caractéristique il y en a au moins *une* qui est une courbe fermée. Mais le système n’étant plus linéaire à coefficients constants, les solutions infiniment voisines ne seront plus homothétiques à cette courbe, mais s’en approcheront asymptotiquement, c’est-à-dire que la solution périodique correspondra à un *cycle limite* de Poincaré, comme l’a fait remarquer M. Andronow. Son amplitude sera ainsi bien déterminée.”

He ended his article by this sentence:

“Je ne puis que renvoyer aux remarquables travaux de cet auteur [Van der Pol], auxquels M. Liénard et M. Andronow ont apporté des compléments fort intéressants.”

As a science historian, Le Corbeiller presented during his various lectures a synthesis of the different results obtained in the field of relaxation oscillations. Moreover, his contribution to the understanding of the processes of development of the non-linear oscillations theory is fundamental since he recalls the essential steps that would surely have fallen into oblivion without his intervention. Nevertheless, it is striking to notice that in his famous lecture given in Paris at the *Conservatoire National des Arts & Métiers* on 6 May and on 7 May 1931, Le Corbeiller [15, p. 4] quoted Andronov very little and never Poincaré:

“Mais c’est un physicien hollandais, M. Balth. van der Pol, qui, par sa théorie des *oscillations de relaxation* (1926) a fait avancer la question d’une manière décisive. Des savants de divers pays travaillent actuellement à élargir la voie qu’il a tracée; de ces contributions, la plus importante nous paraît être celle de M. Liénard (1928). Des recherches mathématiques fort intéressantes sont poursuivies par M. Andronow, de Moscou.”

On 22 April 1932, Le Corbeiller gave a lecture in Paris at the École Supérieure des Postes et Télégraphes where he has been a student many years before (ESPT, 1914). Let’s recall that Poincaré taught in this school from 1902 till 1912 where he also gave many lectures and in particular his ‘forgotten lectures’ on wireless telegraphy.

Discussing the graphical integration of the triode oscillator proposed by Van der Pol [33, p. 983] (see Figure 5), Le Corbeiller [16, pp. 708–709] wrote:

“La théorie générale de ces courbes intégrales fermée, ou *cycles limites*, a été faite par H. Poincaré (2) [1]; la démonstration de l’existence d’un cycle limite unique, dans ce cas actuel, est due à M. Liénard [16].

(2) D’une manière tout à fait générale, l’équation

$$\frac{dx}{X(x,y)} = \frac{dy}{Y(x,y)}$$

équivaut aux deux suivantes:

$$\frac{d^2x}{dt^2} - y\left(\frac{Y}{X} + x\right) + x = 0, \; y = \frac{dx}{dt}.$$

Le mémoire cité de Poincaré équivaut donc à l’étude du système oscillant conservatif $d^2x/dt^2 + x = 0$, soumis à des forces de dissipation et d’entretien dont la résistance est une force *quelconque* de x et de dx/dt :

$$F\left(x, \frac{dx}{dt}\right) = y\frac{Y}{X} + x.$$

[1] H. Poincaré, Mémoire sur les courbes définies par une équation différentielle, Deuxième partie, *Journal de Mathém. Pures et app.* 8, 251, 1882 ; et *Œuvres*, T. 1, p. 44.

[16] A. Liénard, Étude des oscillations entretenues, *Rev. gén. d’électr.* 901 et 946, 1926.”

It is very interesting to remark that Le Corbeiller made reference to the original paper of Poincaré and that he also gave many mathematical details on the way of writing the differential equation characterizing the oscillations of a dissipative system.

The Liénard riddle (1931). As recalled above, in his first paper entitled ‘Étude des oscillations entretenues’, Liénard [17] proved the existence and uniqueness of a periodic solution of a generalized Van der Pol’s equation without making any connection with Poincaré’s works. Then, less than one year later the presentation of Andronov’s notes at the *Comptes Rendus*, Liénard participated with Le Corbeiller and Van der Pol in the *Third International Congress of Applied Mechanics* held in Stockholm from 24 to 29 August 1930 where he presented an article entitled ‘Oscillations auto-entretenues’. According to the title, one might have been thinking that, in this second and last publication on this subject, Liénard would have taken account of Andronov’s result and that he would have established a connection between the periodic solution and a Poincaré’s limit cycle. But, surprisingly not.

In this work, Liénard [18] first summarized his previous results and then he generalized another result established by Andronov and Witt [4] in their second and last note at the *Comptes Rendus* in which they studied the ‘Lyapunov stability’ of the periodic solution, i.e. the stability of a *limit cycle* or ‘orbital stability’. In order to extend Andronov and Witt’s proposition, Liénard [18, p. 176] made use of ‘variational equations’, i.e. of a method introduced by Poincaré [26, p. 162] in the first volume of his famous ‘Méthodes Nouvelles de la Mécanique Céleste’ and which corresponds to what is today known under the name of the computation of ‘characteristics exponents’. To do that, he modified his own equation (6) and replaced it by the following which is now know as ‘Liénard’s equation’:

$$\frac{d^2x}{dt^2} + \omega f\left(x, \frac{dx}{dt}\right) + \omega^2 x = 0. \qquad (8)$$

Then, he wrote:

“Si l’équation (8) admet une solution périodique, de période T, la condition pour que cette solution soit stable est que l’intégrale pendant une période de $\frac{\partial f(x,x')}{\partial x'}dt$ soit positive. La proposition, établie par Messieurs Andronow et Witt [Liénard quotes Andronov and Witt [4]] dans le cas particulier où la fonction $f(x,x')$ est très petite se généralise immédiatement.”

Thus, Liénard [18] generalizes the result of Andronov and Witt [4] for the stability of a periodic solution, i.e. of a limit cycle according to Poincaré’s method of ‘characteristics exponents’ but without quoting Poincaré’s works and without using the terminology ‘limit cycle’ for describing the stable periodic solution (for more details see [10, p. 210]). However, this expression appears in the very first pages of the article of Andronov and Witt [4, p. 256] in footnote:

"(2) Pour la définition des auto-oscillations et la discussion du cas d'un degré de liberté voir A. Andronow, *Les cycle limites de Poincaré et la théorie des oscillations auto-entretenues* (*Comptes Rendus* 189, 1929, p. 559)."

Therefore, it seems very difficult to explain the attitude of Liénard especially since at the first International Conference on Nonlinear Oscillations, to which he was invited, the question of periodic solutions of type limit cycle has been much discussed.

The first 'lost' International Conference on Non-linear Oscillations (1933). From 28 to 30 January 1933 the first International Conference of Non-linear Oscillations was held at the Institut Henri Poincaré (Paris) organized at the initiative of the Dutch physicist Balthasar Van der Pol and of the Russian mathematician Nikolaï Dmitrievich Papaleksi. The discovery of this event, of which virtually no trace remains, was made possible thanks to the report written by Papaleksi at his return in USSR. This document has revealed, on the one hand, the list of participants which included French mathematicians: Alfred Liénard, Élie and Henri Cartan, Henri Abraham, Eugène Bloch, Léon Brillouin, Yves Rocard, ... and, on the other hand the content of presentations and discussions. The analysis of the minutes of this conference highlights the role and involvement of the French scientific community in the development of the theory of non-linear oscillations (for more details see [10, 12]).

According to Papaleksi [23, p. 211], during his talk, Liénard recalled the main results of his study on sustained oscillations:

"Starting from its graphical method for constructing integral curves of differential equations, he deduced the conditions that must satisfy the non-linear characteristic of the system in order to have periodic oscillations, that is to say for that the integral curve to be a closed curve, i.e. a limit cycle."

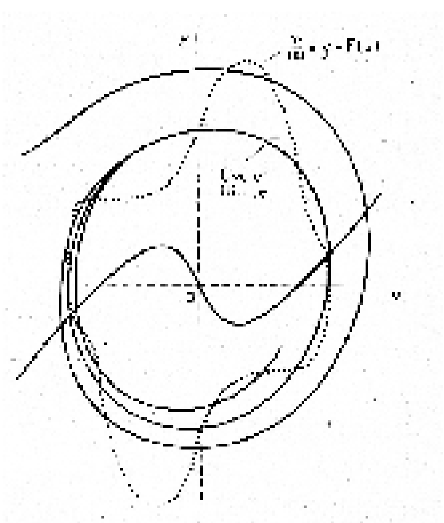

Figure 6 Limit cycle of relaxation oscillator, Rocard [37, p. 220]

This statement on Liénard must be considered with great caution. Indeed, one must keep in mind that Papaleksi had an excellent understanding of the work of Andronov [3] and his report was also intended for members of the Academy of the USSR to which he must justify his presence in France at this conference in order to show the important diffusion of the Soviet work in Europe. Despite the presence of Cartan, Lienard, Le Corbeiller and Rocard it does not appear that this conference has generated, for these scientists, a renewed interest in the problem of sustained oscillations and limit cycles. However, although the theory of non-linear oscillations does not seem to be a research priority in France at that time, it is the subject of several PhD theses and monographs discussed below.

The French PhD theses. During the period 1936–1943 several PhD theses were defended in France on a subject strongly related to non-linear oscillations. Two of them are briefly recalled below (four PhD theses have been found during this period and completely analysed by Ginoux [10]).

The first thesis is that of R. Morched-Zadeh who defended a PhD thesis at the Faculté des Sciences de l'Université de Toulouse in October 1936 entitled 'Étude des oscillations de relaxation et des différents modes d'oscillations d'un circuit comprenant une lampe néon'. (No biographic information could be found concerning this student except the fact that he was Iranian but not parent with Lotfi Morched-Zadeh.) In the introduction of his study Morched-Zadeh [22, p. 3] wrote:

"Au point de vue théorique, les cycles limites de H. Poincaré prennent une grande place dans la théorie des oscillations autoentretenues comme l'ont démontré A. Andronow et A. Witt."

In this case it is very surprising to observe that Morched-Zadeh made reference to the second and last note of Andronov and Witt [4] at the *Comptes Rendus* and not to the first which seemed to be better known and most quoted.

The aim of his work is an experimental study of relaxation oscillations of a neon lamp submitted to various oscillating regimes including of course the case of self-sustained oscillations. This led him to take the very first pictures of a *limit cycle* on a cathode ray tube oscilloscope (see Figure 7).

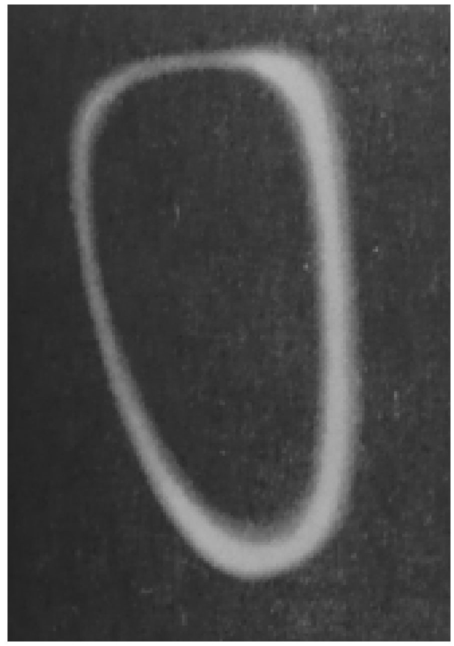

Figure 7 Limit cycle of the neon lamp oscillator, Morched-Zadeh [22, p. 127]

The second thesis is that of Jean Abelé (1886–1961) who was physicist, philosopher and writer (for biographic details see for example [10, p. 325]). He defended his PhD thesis entitled 'Étude d'un système oscillant entretenu à amplitude autostabilisée et application à l'entretien d'un pendule élastique' at the Faculté des Sciences de l'Université de Paris in front of a jury including Yves Rocard. In the introduction of his work Abelé [1, p. 18] wrote:

"À un mouvement périodique *stable* correspond une courbe intégrale fermée *dont s'approchent asymptotiquement en spirales, de l'intérieur et de l'extérieur, pour t croissant, les solutions voisines*. Un des problèmes fondamentaux de la théorie non linéaire consiste dans la recherche de ces courbes fermées, dites *cycles limites*."

Although, the definition of a stable limit cycle exactly corresponds to that given by Poincaré himself, Abelé quotes Andronov [3].

Rocard's textbooks. During World War II, the physicist Yves Rocard (1903–1992) published two manuscripts. Although the title of the first one ('Théorie des Oscillateurs') is very close to that of Andronov and Khaikin [5] published in Russian ('Teoriya kolebaniĭ', 'Theory of oscillations'), the content is quite different. Rocard [36] proposes a synthesis of several works done in this area as well as a summary of the article of Van der Pol [33] on relaxation oscillations with figures including Figure 5 that he commented thus:

"On voit au fur et à mesure que ε croît, se déformer le cycle limite et apparaître les harmoniques."

This is the single occurrence of the terminology limit cycle in the whole textbook, which is given without any reference. (The question of the absence of references in Rocard's book has been much discussed. Interviewed on this issue, Rocard said that because of the war he was unable to access these documents. In fact it has been shown in Ginoux [10, p. 263] that it was not true.)

In 1943, Rocard [37] published his *Dynamique Générale des Vibrations* which has been wrongly considered as a textbook on non-linear oscillations. In fact, in this book which comprises sixteen chapters, only three deal with this subject. In chapter XV, Rocard [37, p. 220] recalls the results of Liénard [17] and plots Figure 6. He explained:

"...on constate (comme autrefois H. Poincaré l'a démontré) que la courbe intégrale s'enroule un certain nombre de fois et tend vers une courbe fermée dite *cycle limite*, qui dans cette représentation correspond au régime d'oscillations permanent."

Here again, there is no reference neither to Poincaré nor to Andronov.

Discussion

This study has shown that two major contributions to the development of non-linear oscillation theory had occurred in France during the first half of the twentieth century. The first is the correspondence between the concept of limit cycles and the existence of a stable regime of sustained oscillations in wireless telegraphy established by Henri Poincaré in 1908 in these 'forgotten lectures' at the École Supérieure des Postes et Télégraphes, and the second is the same kind of correspondence, established twenty years later in a more general context by the Russian mathematician Aleksandr Andronov in his famous note at the *Comptes Rendus*.

If the 'discovery' of these 'forgotten lectures' demonstrated that Poincaré has stated that the periodic solution of a non-linear differential equation characterizing the non-linear oscillations of a particular radiophysics device named *singing arc* is nothing else but a stable limit cycle, it has really produced no reaction on the French scientific community.

Nevertheless, the analysis of the influence of Andronov's note on this scientific community from 1929 to 1943 has shown that it was nearly the same. Liénard, for unknown reasons didn't make use of the terminology 'limit cycle' neither before 1929 nor after. Moreover, although he became probably aware of Andronov's correspondence during the first 'lost' International Conference on Non-linear Oscillations in 1933 he has not pursued his research in this area. But, in 1933, Liénard was 64 years old and near to retirement. This was not the case for Le Corbeiller who has been one of the first to establish a deep connection with Poincaré's works. However, when World War II was declared he went to the USA and became a Professor at Harvard. Concerning Rocard, who made only few allusions to Poincaré's concept of limit cycle in his textbooks, he turned to nuclear research immediately after World War II.

Thus, it seems that although France has been a kind of crossroads for the development of non-linear oscillations theory, nobody has succeeded in unifying French scientists around a research program in this area.

In fact, the mathematician Jacques Hadamard has been deeply involved in this task at various levels. First, he has presented during the 1930's many notes at the French Academy of Sciences on this subject coming from the USSR: that of Andronov, of Andronov and Witt but also that of Kryloff and Bogoliouboff. But, he has also presented the works of Poincaré during his seminar at the *Collège de France* from 1919 to 1932. Unfortunately it didn't produce any reaction from the French scientific community.

So, although this community has produced many fundamental results necessary for the development of the non-linear oscillation theory such as that of Liénard for example, there has been no research program like in the USSR or in the USA and so, no 'School of non-linear' in France. The terrible impact of the two world wars is probably responsible for such a lack of organization.

During the commemoration of the centenary of Poincaré's birth, the newspaper *Le Monde* published on 15 May 1954 the article entitled 'Les conferences de Henri Poincaré à l'École Supérieure des P.T.T.' in which we learn that Eugène Reynaud-Bonin, a former student of this school (ESPT, 1911), has attended to Poincaré's 'forgotten lectures' in

Figure 8 *Le Monde*, 15 May 1954

1908 and 1910. His interview was reproduced in *Le Monde* (see Figure 8).

Three days later, the physicist Nicolas Minorsky [21] presented a lecture to civil engineers entitled 'Influence d'Henri Poincaré sur l'évolution moderne de la théorie des oscillations non linéaires' in which he began with these words:

"La répercussion des travaux d'Henri Poincaré s'est fait sentir dans presque tous les domaines des sciences appliquées, mais c'est surtout dans la théorie des oscillations qu'elle a provoqué de tels changements que cette théorie est aujourd'hui passablement différente de ce qu'elle soit."

Then, he concluded with this sentence which shows that he didn't have knowledge of Poincaré's 'forgotten lectures':

"Il est difficile de trouver dans l'histoire de la Science un autre exemple de théorie mathématique développée sans aucune relation aux applications ... qui ait présenté une base aussi parfaite pour l'étude des phénomènes innombrables qui se sont révélés depuis lors, sans qu'il y ait presque rien a changer à cette théorie un demi-siècle plus tard."

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Poincaré and the idea of a group

In many different fields of mathematics and physics Poincaré found many uses for the idea of a group, but not for group theory. He used the idea in his work on automorphic functions, in number theory, in his epistemology, Lie theory (on the so-called Campbell–Baker–Hausdorff and Poincaré–Birkhoff–Witt theorems), in physics (where he introduced the Lorentz group), in his study of the domains of complex functions of several variables, and in his pioneering study of 3-manifolds. However, as a general rule, he seldom appealed to deep results in group theory, and developed no more structural analysis of any group than was necessary to solve a problem. It was usually enough for him that there is a group, or that there are different groups. In this article Jeremy Gray gives a brief history on Poincaré's group idea.

It is well-known that between 1880 and 1884 Poincaré brought together in a completely unexpected way the subjects of complex function theory, linear differential equations, Riemann surfaces, and non-Euclidean geometry (see, for example, [42] and [7]). Cauchy's approach to complex function theory and the theory of differential equations were mainstream topics in the education of a French mathematician at the time, but Riemann surfaces were not, largely because Riemann's way of thinking was not congenial to Charles Hermite, who dominated the scene in the 1870s.

In the spring of 1879 the Académie des Sciences in Paris announced the topic of the Grand prize for the mathematical sciences (see *C.R. Acad. Sci.*, 88, 1879, p. 511), which was "to improve in some important way the theory of linear differential equations in a single independent variable". The topic had been proposed by Hermite, and his intention was to draw young French mathematicians to study the work of Lazarus Fuchs, who was the German expert on the subject; in the aftermath of the Franco-Prussian War catching up with the Germans was on every patriotic Frenchman's mind.

Automorphic functions

In a series of papers in 1866 to 1868, starting with [6], Fuchs had been able to characterise a class of linear ordinary differential equations of arbitrary order that have the property that their solutions are meromorphic and have poles only where the coefficients of the equation themselves have poles. Among this class is the celebrated hypergeometric equation,

$$z(z-1)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\,\frac{dw}{dz} - abw = 0 \tag{1}$$

and we may confine our attention to it, although by then Fuchs had moved on to study other problems.

Poincaré took from the later work of Fuchs the idea that the quotient of a basis of solutions to equation (1) was an interesting object. If the solutions are denoted $w_1(z)$ and $w_2(z)$, and the quotient as $\zeta(z) = \frac{w_1(z)}{w_2(z)}$, then analytic continuation of the solutions around a path enclosing a singular point returns the quotient in the form

$$\zeta(z) = \frac{a_{11}w_1(z)+a_{12}w_2(z)}{a_{21}w_1(z)+a_{22}w_2(z)} = \frac{a_{11}\zeta(z)+a_{12}}{a_{21}\zeta(z)+a_{22}},$$

where the coefficients a_{jk} are constants that depend on the path. As this formula makes clear, the 'function' ζ is a multi-valued function, but its set-theoretic inverse is a generalisation of a periodic function:

$$z(\zeta) = z\left(\frac{a_{11}\zeta+a_{12}}{a_{21}\zeta+a_{22}}\right). \tag{2}$$

Geometrically, the function $\zeta(z)$ maps the upper half-plane to a triangle, the vertices of which are the images of the singular points $0, 1, \infty$ on the real axis, and the angles of which are determined by the coefficients a, b, c of the hypergeometric equation. Analytic continuation shows that the lower half-plane is mapped to another triangle (which will be congruent if the coefficients are all real) and thereafter the images of the half-planes form a net of triangles, provided the angles are of the form π/n for some integers n. Moreover, each triangle will have the same three angles, although their sides will be circular arcs and not straight, and Poincaré introduced a simple geometric transformation to straighten them out. This much, but little more, formed the content of the essay Poincaré submitted for the prize in May 1880. The extra concerned a discussion of the net of triangles and made some corrections to Fuchs's papers. He showed, for example, that if the angles of the triangles are $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{6}$, then the net covers the plane, eight suitably chosen triangles form a parallelogram, and $z = z(\zeta)$ is an elliptic function; but if the angles are $\frac{\pi}{4}$ and $\frac{\pi}{2}$ at the image of $z = 0$ and $z = 1$ respectively and $\frac{\pi}{6}$ at ∞, then the net lies inside a certain circle, and the sides of the triangles are circular arcs meeting this circle at right angles.

Fuchsian functions

Poincaré then wrote to Fuchs, and it became obvious that he had a much clearer idea of this net of triangles, which he was beginning to think of as the domain of the inverse function $z = \zeta(z)$, than did Fuchs. The correspondence was very amicable, all the same, and on 12 June 1880 Poincaré wrote to Fuchs to say: "I have found some remarkable properties of the functions you define, and which I intend to publish. I ask your permission to give them the name of Fuchsian functions." Fuchs, of course, agreed. (The correspondence between Poincaré and Fuchs is reproduced in Poincaré's, *Oeuvres* 11, pp. 13–25.) But at this point Poincaré was in fact stuck on the case of second-order linear differential equations with no more than three singular points. What happened next was recalled by Poincaré twenty-eight years later when he addressed the Société de Psychologie in Paris about his experience of mathematical discovery. He explained in the lecture [39, *Science et Méthode*, p.51] that one night coffee had kept him awake, and ideas surged up in his mind, eventually forming stable combinations until: "In the morning I had established the existence of a class of Fuchsian functions, those which are derived from the hypergeometric series. I had only to write up the results, which just took me a few hours." He then followed the analogy with elliptic functions and created his thetafuchsian series. The restless night could have been between 29 May and 12 June, when he discovered the Fuchsian functions, or between 12 and 19 June 1880, by which time he knew much more about them.

Next came the realisation that, more than anything else in this part of his work, was to make Poincaré's name among mathematicians [39, *Science et Méthode*, pp. 51–52], when he realised, as he boarded a bus on a geological expedition organized by the École des Mines, that "the transformations I had made use of to define the Fuchsian functions were identical with those of non-Euclidean geometry". If the bus trip took place before the 12th the two breakthroughs happened in a rush, which is not quite the impression Poincaré's account suggests, so perhaps it happened between the 12th and the 19th.

The realisation on boarding the bus was surely that his simple geometric transformation converts the pictures of a net inside a disc made of triangles with circular-arc sides into a net of triangles inside a disc but with straight sides — and this is the non-Euclidean disc of Beltrami. It follows that the original net of triangles can be regarded as made up of congruent copies of the same triangle, where congruence is to be taken with its non-Euclidean meaning. Now instead of analytic continuation as the basic mechanism, Poincaré had isometries to work with. We do not know how Poincaré first heard of non-Euclidean geometry, but Hoüel had translated the original papers by Bolyai

Lazarus I. Fuchs (1833–1902)

and Lobachevskii into French in the 1860s and Helmholtz had also written on the subject.

On 28 June 1880 Poincaré sent a 70-page supplement to his entry to the Académie des Sciences. In it he regarded successive analytic continuations as rotations, denoted M and N, of the basic triangle, and the role of non-Euclidean geometry [42, p. 35]. Significantly, he defined the geometry through its group, writing that "the group of operations formed by means of M and N is isomorphic to a group contained in the pseudogeometric group. To study the group of operations formed by means of M and N is therefore *to do the geometry of Lobachevskii*". He then described the basic features of this convenient language of non-Euclidean geometry, defining points, lines, angles, and equality of figures — two figures are equal if one is obtained from another by a non-Euclidean transformation. Then he turned to the study of the Fuchsian functions, remarking that "the Fuchsian function is to the geometry of Lobachevskii what the doubly periodic function is to that of Euclid", but at this stage he was unable to establish the convergence theorems for the functions.

Poincaré was still able to work only with triangles until at least 30 July, but at some date in August he was rescued by his earlier work on arithmetic. As he recalled in his lecture in 1908, [39, *Science et Méthode*, pp. 52–53], while walking on the cliffs he realised that "the arithmetical transformations of ternary indefinite quadratic forms were identical with those of non-Euclidean geometry." We shall see below that ternary indefinite quadratic forms (objects of the form $x^2 + y^2 - z^2$) were exactly what he had been studying in his papers on the consequences of Hermite's number theory.

The uniformisation theorem

Poincaré now wrote another supplement to his essay, describing the polygonal case, and making progress with the convergence arguments, and he followed it with a third supplement before the competition closed. Even so he did not win, the prize went to Georges Halphen. But by 1881 he had a lot to publish, and as he began to do so more and more ideas occurred to him. This work brought him international attention of two very different kinds. The young Swedish mathematician Gösta Mittag-Leffler was in the process of setting up a new mathematical journal, to be called *Acta Mathematica*, and he saw immediately that Poincaré's papers would establish his journal as one of international significance. The young German mathematician Felix Klein saw someone entering territory he had staked out as his own, although he did not perhaps foresee at first what a challenge this might be.

Both men entered into a correspondence with Poincaré. Mittag-Leffler got the long papers he wanted in which Poincaré set out the theory of the new functions, and Klein had to realise that the younger, less well-educated Poincaré was moving faster than he ever could. Their exchanges are both scholarly and personal. Klein objected to the name 'Fuchsian' for the new functions on the grounds that some ideas of Schwarz were much closer, and Poincaré agreed when he got round to consulting Schwarz's paper, which he had not known. But he could not agree to change the name, which he had already used in publications, and Klein railed against this, doubtless because Fuchs, as a Berlin-trained mathematician close to Weierstrass and Kummer, was a rival likely to have a better career than him but with less talent. To shut him up Poincaré named the generalisation of Fuchsian functions that require 3-dimensional non-Euclidean geometry 'Kleinian functions'. Klein protested, correctly, that he had had nothing to do with these functions and Schottky's name would be more appropriate; "Name ist Schall und Rauch" Poincaré replied in German ("Name is sound and fury", the quotation comes from Gretchen in Goethe's *Faust*).

The correspondence was also a competition, and it was to cost Klein his health, but not before both men had come to the deepest jewel in the field, what became known as the uniformisation theorem. Klein took the lead, because he knew Riemann's theory of the moduli of algebraic curves on Riemann surfaces according to which a Riemann surface of genus $g > 1$ depends on $3g - 3$ complex parameters. On the other hand, it was possible to show that the non-Euclidean polygons that are mapped around in the disc by a Fuchsian group in such a way that the quotient space is a Riemann surface of genus g also depend on $3g - 3$ complex parameters. The implication was obvious: every Riemann surface arises as a quotient of the disc with its non-Euclidean metric under the action of a Fuchsian group. Not only that, but locally the map from the disc to the Riemann surface will be an isometry, so all but the Riemann surfaces of lowest genus locally carry non-Euclidean geometry. (This copies the case of elliptic functions, which are quotient spaces of the Euclidean plane.) Klein's formulation of this result [14, 15] was perhaps sharper than Poincaré's, but neither man come close to a proof; Poincaré's account [24], generalised to apply to the uniformisation of any many-valued function (not necessarily algebraic), was particularly obscure. The topic was made one of Hilbert's problems in his address at the Paris ICM in 1900 (see [13]), and the first proofs were given independently by Koebe and Poincaré in 1907. (See [38] and Koebe's paper [16] and his many subsequent publications.) Poincaré's many notes and his long papers in *Acta Mathematica* exploited the analogy with elliptic functions. In [22] he defined a Fuchsian group to be the one associated to a polygon with angles of the form π/n where n is an integer, which identifies the edges of the polygon in pairs, and where the edges are arcs of circles perpendicular to the boundary of a circle. In [23] he turned to the task of defining automorphic functions.

New functions (see [23]) were defined by using sums of the form

$$F(z) = \sum_{\gamma \in G} f(\gamma z),$$

where the summation is taken over all the elements of a Fuchsian group G. This would ensure that the function F is G-invariant, but first Poincaré had to confront several problems: the sums had to be convergent, and it was to turn out that non-trivial examples could converge to the zero function. To work round this problem he generalised the idea of theta functions, and introduced functions of the form

$$\theta\left(\frac{az+b}{cz+d}\right) = \theta(z)(cz+d)^m, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

where m is an integer. He established convergence for $m > 2$ by an ingenious argument mixing Euclidean and non-Euclidean geometry, and showed that products of quotients of these 'thetafuchsian' functions were Fuchsian functions. He then showed that the inverse of a Fuchsian function is a quotient of two functions that satisfy a second-order linear differential equation that was intimately connected to the polygon.

What is very striking is that Poincaré offered a very general theory, with a small number of examples to illustrate how local difficulties can be tackled. He did not offer a detailed presentation of Fuchsian groups; if a reader wanted to know what Fuchsian groups were like and if in particular there were any well-known examples in the literature the best that could be said is that Poincaré left the field wide open. This is another way in which he resembled Riemann, and it was to be the pattern throughout his work in mathematics.

Epistemology

In 1891 Poincaré wrote the first of the many popular essays, [27], that were to keep him before a large audience thereafter. He had become famous in 1889 for winning the prize competition organised by King Oscar II of Sweden for his essay, [26], on the stability of the solar system. (For a rich history of this event, see [2].) The success also saw him appointed a Knight of the Legion d'Honnneur, and after a short essay on what he had done on celestial mechanics the next popular essay was a rather cryptic account of geometry. Here again it became clear that he put the group ahead of the space, or, as he preferred to say, the form ahead of the matter. This made him a person with an eagerly sought-after opinion on the burning question of the day: Is space Euclidean or non-Euclidean? His surprising answer was that one could never tell. In his view both geometries were consistent, and he set out a dictionary of terms for translating from one geometry to the other. This showed that any contradiction in one geometry would provide a contradiction in the other, and so the two were relatively consistent (he admitted this did not establish the consistency of either but said he had some ideas of how this could be done).

He asked his readers to imagine some experiment in which a seemingly decisive result had been obtained, for example the construction of a figure with light rays marking out four equal sides meeting at four equal angles for which the sum of the angles was less than 2π. This would seem to suggest that space was non-Euclidean, but, said, Poincaré, there is another interpretation, which was that space was Euclidean and light rays were curved. There could be no way of deciding logically between these two interpretations, and all we could do would be to settle for the geometry we found most convenient, which, indeed, he said would be the Euclidean one. His reasons were, however, unexpected, and will be considered shortly.

Not long after this, Poincaré was drawn into a long dispute with Russell, who began with the intention of establishing on Kantian lines that we must be able to form some idea of geometry, else we could not have a concept of space, and that, furthermore, that concept of geometry was necessarily projective geometry. Once this was established, Russell then sought to graft a concept of distance onto this framework, but he did not assume that space was Euclidean (see [44] and [45]).

Poincaré, in his [30], had little trouble demolishing the clumsy presentation of projective geometry that Russell offered, and Russell endeared himself to Poincaré by his willingness to concede his errors in print (this ability was to remain a charming feature of all Russell's philosophical investigations). But the further the debate went on the clearer it became that the two had fundamentally different starting points. To Russell it was clear "before we begin", as he put it, that the distance from London to Paris is greater than a millimetre. But to Poincaré this was not how one could begin.

The best account he gave of his philosophy he published in an English translation in the *Monist* [29] in 1898. He began by raising the fundamental question of how we construct a sense of space around us at all. This was much discussed by psychologists at the time, and Poincaré observed that we can construct many spaces. A single motionless eye would construct a two-dimensional projective geometry with no sense of distance. A pair of eyes could construct a sense of depth. We have our sense of touch, and we could construct a high-dimensional space by recording the muscular sensations needed to put the tip of a finger somewhere. Out of this welter of experiences and before we are capable of formal instruction, we all construct a sense of three-dimensional space occupied by some bodies with predictable behaviour. These are the rigid bodies, and they are singled out by the fact that we can compensate for the motion of a rigid body by a motion of our own. We can, for example, distinguish the motion of a glass from the motion of the wine swirling around in it. So, he argued, we build up in our minds a sense of what rigid bodies are, and are able to handle them hypothetically. This mental construction gives us our idea of the isometries of a hypothetical rigid body, and from this we construct our concept of space – notice that rigid bodies and their motions came first in this analysis. The concept of distance is derived from the behaviour of rigid bodies, which is why on this account Poincaré could dispute Russell's claim about London, Paris, and the millimetre. For Poincaré the claim is true not because we know what distance in space is, but because we know what isometries are. This knowledge is innate, it has evolved with the human species, and it is triggered by the experiences of every sentient infant.

For Poincaré it is only via the introduction of the group that the non-measurable 'space' of Helmholtz and Lie becomes a measurable magnitude "that is to say, a veritable space". Therefore [29, Sections 21 and 22]:

"What we call a geometry is nothing but the formal properties of a certain continuous group . . . so that we may say, space is a group. But the assertion is no less true of the notion of many other continuous groups; for example, that which corresponds to the geometry of Lobachevskii. There are, accordingly, several geometries possible, and it remains to be seen how a choice is made between them."

Poincaré was adamant that different creatures, with a different history, might be non-Euclidean in the sense that their brains would find non-Euclidean geometry convenient. If we met such creatures, we would not share their sense of what is easy of natural, nor would they share ours, but neither side would be able to trap the other in a contradiction. Where we and they would differ would be in our innate understandings of rigid bodies, and we would be the ones whose brains appreciated that the translations in the group of isometries formed a normal subgroup, a statement that is false in the group of non-Euclidean isometries.

This is fundamental epistemology (and quite different from his other conventionalisms). It accounts for the one wry observation Poincaré had on Hilbert's foundations of geometry as they were set out in his *Grundlagen der Geometrie* [12], in 1899. Hilbert gave a purely axiomatic formulation, with no pretence to being an account of how we can have knowledge of the external world, and when Poincaré reviewed them he commented [33, p. 272] that:

"The objects which he calls points, straight lines, or planes become thus purely logical entities which it is impossible to represent to ourselves. We should not know how to picture them as sensory images . . . Each of his geometries is still the study of a group. The logical point of view alone appears to interest him. With the foundation of the first proposition, with its psychological origin, he does not concern himself. His work is then incomplete; but this is not a criticism . . . Incomplete one must indeed resign one's self to be."

Groups in number theory

When Poincaré began to publish in 1880 it was not only on linear differential equations in the complex domain (as discussed above) and on the curves defined by real differential equations, but on the number theory of forms in several variables, as well as some other papers on number theory, and the subject remained a lifelong interest of his. His second paper, [21], picked up on a remark of Hermite's that there is a group of transformations mapping an indefinite ternary quadratic form to itself. Poincaré showed how these transformations can be regarded as isometries of the non-Euclidean disc, and then how this illuminates the reduction of these forms to canonical form.

The most significant of Poincaré's papers on number theory were on Fuchsian functions and arithmetic, and grew out of his previous work on ternary forms. In [25] he observed that the linear transformations that map the ternary quadratic form $\Phi = y^2 - xz$ to itself have an eigenvalue $+1$ or -1, and those with eigenvalue $+1$ can all be written in the form

$$\begin{pmatrix} \delta^2 & -\delta\gamma & \gamma^2 \\ -2\delta\beta & \alpha\delta+\beta\gamma & -2\alpha\gamma \\ \beta^2 & -\alpha\beta & \alpha^2 \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are four arbitrary quantities such that $(\alpha\delta - \beta\gamma)^2 = 1$. Poincaré restricted his attention to the case when they are all real and moreover $\alpha\delta - \beta\gamma = 1$. Moreover, the groups of Fuchsian transformations

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta},$$

and of 3 by 3 matrices defined above are isomorphic.

If, however, T is a linear transformation of the variables x, y, z that does not map the form Φ to itself, then the new form, which Poincaré denoted $F = \Phi T$, that is mapped to itself by all the transformations of the form $\Sigma = T^{-1}ST$, and so the group of self-transformations of F and of Φ are isomorphic (indeed, conjugate).

On the further assumption that the coefficients of the form F are integers and that the coefficients of the self-transformations of F are likewise integers, number theorists such as Hermite had already been led, he remarked, to interesting discontinuous groups. Poincaré had earlier called the corresponding Fuchsian functions 'arithmetic Fuchsian functions', and he now proposed to show that these functions satisfied a theorem generalising the addition theorem for elliptic functions, which more general Fuchsian functions do not.

To establish this result, Poincaré examined the Fuchsian groups that are associated to different (ternary quadratic) forms, and divided them into four families according as they have no elliptic or parabolic elements, elliptic but no parabolic elements, parabolic but no elliptic elements, or both elliptic and parabolic elements. He showed that it was always possible to determine which of these four cases any given ternary form belonged to. He then looked at the geometry of the corresponding Fuchsian polygon — the number of its sides, the size of its angles, whether the vertices lie inside or on the boundary of the non-Euclidean disc — and showed that in each case this information was determined by the quadratic form.

After a consideration of the geometry of the Fuchsian polygons and its connection to the arithmetic properties of the form F, Poincaré turned to the classical example of the modular function $J(z)$, which is invariant under the group $PSL(2,\mathbb{Z})$. The transformation S given by $z \mapsto z/n$ is not in this group, and the relationship between $J(z)$ and $J(z/n)$ is governed by the celebrated modular equation. Poincaré noted that the groups Γ and $S^{-1}\Gamma S$ are what he called commensurable, that is to say their intersection is of finite index in each of them. Indeed, the elements of $S^{-1}\Gamma S$ are of the form

$$z \mapsto \frac{\alpha z + \beta/n}{\gamma n z + \delta},$$

where $\alpha, \beta, \gamma, \delta$ are integers and $\alpha\delta - \beta\gamma = 1$. The required subgroup of Γ and $S^{-1}\Gamma S$ consisted of elements of the form

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

where $\gamma \equiv 0 \bmod n$. A repeat of the same argument showed that $J(z)$ and $J(pz/n)$ are algebraically related, and more generally that $J(z)$ and $J\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$ are algebraically related.

To generalise the modular function, Poincaré returned to the three groups that map a given ternary form to itself: those where the corresponding matrices have real, rational, or integer coefficients. Each of these group gives rise to a corresponding Fuchsian group; when the coefficients are integers Poincaré called the corresponding Fuchsian group the principal group. When, however, one starts from the group of matrices with rational coefficients that preserved a given form, the corresponding Fuchsian group is not discontinuous, and Poincaré chose an element S in it with rational, non-integral coefficients. This element gave rise to a Fuchsian transformation s that was not in the principal Fuchsian group, and by his earlier results this means that a Fuchsian function for the principal group is algebraically related to its transform by s. Poincaré continued on this theme in [35], which he extended and corrected in [41] (as a footnote observes, Poincaré posted this paper 7 July 1912, the day he went in to hospital for his fatal operation).

Poincaré and Lie theory

Helmholtz was the first person to draw attention to the importance of rigid-body motions in geometry, and there is no doubt that his ideas influenced Poincaré. But his investigation of the nature of the corresponding groups, the types of elements they could possess, and the number of points of space that could be fixed before the group element was also fixed was not mathematically rigorous, and Felix Klein asked Sophus Lie to improve it, which he did in the third volume of his *Theorie der Transformationsgruppen*, 1893. This, and all of his other investigations, impressed Poincaré, who had a high opinion of Lie and assisted in the effort to send promising students to work with him so that his ideas could be written up. But Poincaré did not work on Lie's ideas himself until after Lie's death in 1899. What he then did has remained obscure and unappreciated until the work of Schmid [46] in 1982, Grivel [9] in 2006, and most recently Achilles and Bonfiglio [1], and can only be briefly noted here. (See these authors for references to the original papers.)

Lie's third fundamental theorem

For Lie, a transformation from a space to itself was typically given by an expression of the form $y = f(x, a_1, \dots, a_n)$, where x and y belong to the space and the a's are parameters. For example, the rotation of a sphere about a fixed axis defines a transformation with the angle of the rotation as the single parameter a. Composition of transformations Lie presumed produced a third transformation of the same type,

$$\begin{aligned} &f(f(x, a_1, \dots, a_n), b_1, \dots, b_n) \\ &\quad = f(x, c_1, \dots, c_n) \end{aligned} \tag{3}$$

where the values of the parameters c are determined by the values of the a's and b's. The composition is valid for rotations of a sphere about an axis, and any pair of parameters determine a third, but in general this is false.

Sophus Lie (1842-1899)

When it is valid Lie said the transformations formed a group (the modern term is a group germ). Lie did not demand inverses, nor indeed did he specify that there be an identity transformation of this form.

Lie considered the transformations obtained as the parameters vary by an arbitrarily small amount. He often thought of an infinitesimal transformation of this kind as a directional derivative, and wrote it in in the form $y = x + t\xi(x)$, where t is arbitrarily small and ξ depends on the parameters a. For, any smooth function φ of the x's, now regarded as k-tuples $x_1, \ldots, x_k$, can be given a Taylor series expansion

$$\varphi(y_1, \ldots, y_k) = \varphi(x_1, \ldots, x_k) + t\left(\xi_1 \frac{\partial \varphi}{\partial x_1} + \cdots + \xi_k \frac{\partial \varphi}{\partial x_k}\right).$$

Lie denoted $\xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_k \frac{\partial}{\partial x_k}$ by X; he called X an infinitesimal operator, and wrote

$$\varphi(y_1, \ldots, y_k) = 1 + tX\varphi(x_1, \ldots, x_k).$$

These infinitesimal operators combine according to rules already studied by Poisson and Jacobi. The composition of two infinitesimal operators X_j and X_k corresponding to the same group germ, written $[X_j, X_k]$, is given by an expression of the form

$$[X_j, X_k] = \sum_l c_{jkl} X_l, \qquad (4)$$

where the c_{jkl} are constants and the X_l run through a basis for the infinitesimal operators corresponding to the same group germ. Lie claimed the converse, that given partial differential operators $X_1, \ldots, X_n$ for which equation (4) holds and for which the constants c_{jkl} must satisfy some trivial identities, there are transformations (3) which have the X_j as their infinitesimal counterparts and a rule for determining the $c_1, \ldots, c_n$ as functions of the $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$, such that equation (3) holds. It is this converse, known as Lie's third fundamental theorem, that attracted the most interest. In more modern language, it concerns the exponential map from a Lie algebra of a Lie group to the corresponding group. It asks: Given $X, Y \in g$, find Z such that $\exp(X).\exp(Y) = \exp(Z)$.

As the above authors show, Lie himself had offered two proofs of his third fundamental theorem, one in the special case when the group has no centre, and one in the language of differential equations. In 1890 Friedrich Schur had given a different proof, which showed that the way the c's depended on the a's and b's was determined by an infinite series of a universal form (independent of the parameters) in which the coefficients were polynomial functions of the a's and b's. In 1901 the Oxford mathematician John Edward Campbell addressed the question in its exponential form directly, and dealt successfully with the formal, algebraic side of the problem but did not deal with the convergence of the power series he exhibited.

Convergence of the power series

In [31] and [32] Poincaré offered his proof, as part of a volume celebrating the work of Sir George Stokes, only to find that his work was very close to that of Campbell and Schur, although he hoped that his ideas had enough in them still to merit publication. He claimed to have reduced the problem to the solution of simple differential equations, a process that could, he said, be carried out in finite terms. His argument is too long and technical to describe here, but unlike Campbell he did tackle the convergence of the power series, making an ingenious use of the residue calculus of complex function theory. He otherwise adhered to a purely formal point of view, abstracting, as he put it, completely from the 'matter' of the group, the same phrase he had used earlier in discussing how we use our innate concept of a group to construct space in his [29], and found formulae applicable to all isomorphic groups. These formulae gave relations between the parameters in the transformations from which he could construct a group isomorphic to the one he was studying, and so establish Lie's third theorem. The same question was then tackled by Felix Hausdorff [11], who noted the existence of a more recent paper by Baker written when he was finishing his own. Hausdorff gave an account that he regarded as a considerable simplification of Poincaré's, as it is, algebraically (convergence is established much as Poincaré had done). Hausdorff's account met with approval of Bourbaki (see [3]), and that is one reason the result is today known as the Campbell–Baker–Hausdorff formula.

Poincaré–Birkhoff–Witt theorem

The other major result in this area to which Poincaré contributed has been known since Cartan and Eilenberg's [5] as the Poincaré–Birkhoff–Witt theorem. Ton-That and Tran, in their thorough account of the history of this theorem [47], concluded that: "But by a careful analysis of Poincaré [31] one must conclude without a shade of doubt that Poincaré had discovered the concept of the universal algebra of a Lie algebra and gave a complete and rigorous proof of the so-called Birkhoff–Witt theorem." Poincaré again displayed his liking for form over matter in proving this result, which is a major result about the structure of a Lie algebra. (See [47] for a thorough account of Poincaré's work and the convoluted history of the names for this theorem.)

It is only possible to be brief here about what Poincaré accomplished. Given a finite-dimensional family of vector fields $X_1, X_2, \ldots, X_n$ or infinitesimal transformations with a multiplication, so

$$[X_j, X_k] = \sum_l c_{jkl} X_l,$$

where the so-called structure constants c_{jkl} satisfy certain identities that follow from the properties the multiplication, Poincaré drew attention to the largest algebra one can usefully construct from these symbols. He formed equivalence classes of these relations, and showed that the set of equivalence classes is the universal enveloping algebra of $\mathcal{L}$. He then gave a detailed analysis if the best form for the representatives of each equivalence class, which was highly symmetrical; the modern statement of this result is the Poincaré–Birkhoff–Witt theorem. It implies that there is a canonical injective map from $\mathcal{L}$ to its universal enveloping algebra, and therefore any Lie algebra over a field is isomorphic to a Lie subalgebra of an associative algebra and some problems about Lie algebras can be transferred to problems about associative algebras.

Poincaré's Lorentz group

Lorentz, in his *Electromagnetic Phenomena* [19] had discussed the failure of attempts to detect the motion of the Earth relative to the ether that could have detected effects of the order of $\frac{v^2}{c^2}$. He explained this null result by the hypothesis of a contraction later named after him, on supposing that bodies contract in the direction of motion of the Earth, thus making any attempt to detect motion relative to the ether impossible. Poincaré, in his short note [34] said that this was so important that he had been inspired to take up the question, and had been able to come to results that agreed in all important respects with those of Lorentz, although he wished to modify and complete them in some respects.

Poincaré did not respond to Lorentz's arguments, but took from him the observation that the equations of the electro-magnetic field were unaltered by transformations of the form

$$x' = kl(x + \varepsilon t),\ y' = ly,\ z' = lz,$$
$$t' = kl(t + \varepsilon x),$$

where x, y, z are the coordinates and t the time before the transformation and x', y', z' and t' afterwards. Furthermore, ε is a constant that depends on the transformation, and $k = 1/\sqrt{1-\varepsilon^2}$. Poincaré insisted that these transformations must form a group, and deduced from this that $l = 1$, a conclusion, he observed, that Lorentz had reached in another way. Poincaré's argument relied on the fundamental principle of relativity: two observers in constant relative velocity must make measurements that differ in a way that only takes note of their different situations (positions and velocities). Poincaré had first explained this argument to Lorentz in a letter in 1904–1905 (see Miller [20, p. 81] for a facsimile of the letter).

The postulate of relativity

In his long paper [36] Poincaré set forth a more elaborate position. He raised the impossibility of detecting motion with respect to the ether to the status of a law of nature and dignified it with the name of the *postulate of relativity*. He agreed that Lorentz transformations do imply the postulate of relativity, but now the new Lorentz contraction must be explained, and the deep problem, said Poincaré, was not that of explaining something, but knowing what has to be explained. The speed of light, said Poincaré, appears in many branches of physics, each time with the same size, if we admit the principle of relativity. This might be because everything in the world is electro-magnetic in origin — a grand claim much discussed at the time — or simply from the way we measure everything, which is based on the assumption that two lengths are equal if it takes light the same time to traverse them. That said, Poincaré also apologised for bringing forward partial results at a time when the whole theory might be in danger from fresh discoveries in the study of cathode rays (i.e., fast-moving electrons).

To study the effect of a Lorentz transformation on electro-magnetic phenomena, Poincaré wrote Maxwell's equations in the potential form, and in units in which $c = 1$. He then observed (see [36, p. 499]) that if in one frame the equation of a moving sphere is

$$(x-\xi t)^2 + (y-\eta t)^2 + (z-\zeta t)^2 = r^2,$$

then the equation in transformed coordinates described a moving ellipsoid, the shape of which depends on the velocity of the sphere. He next obtained the components of the electric and magnetic fields in the new coordinate frame, noting that his results agreed with those of Lorentz, and observed that Maxwell's equations were still satisfied. He also wrote down the new addition law for velocities [36, p. 500]: if two electrons have velocities ξ and ξ' relative to an observer, one has a velocity of $\frac{\xi+\xi'}{1+\xi\xi'}$ relative to the other. Finally he returned to the problem of the shape of a moving electron and the stresses required to ensure that the electron is in equilibrium. (See [20, p. 55 et seq.] for a discussion of this.)

Poincaré then re-obtained the invariance of Maxwell's equations under Lorentz transformations, and in Section 4 of his paper Poincaré showed that the Lorentz transformations form a group provided that $l = 1$. His argument is notable for using Lie's method of infinitesimal generators. Poincaré began by considering transformations of the form

$$x' = kl(x + \varepsilon t),\ y' = ly,\ z' = lz,$$
$$t' = kl(t + \varepsilon x), \tag{5}$$

where $k^{-2} = 1 - \varepsilon^2$. He wrote down the composition law, and then found the infinitesimal transformations that generate the group. He argued that when $l = 1$ and ε is infinitely small, there is a transformation, T_1,

$$\delta x = \varepsilon t,\ \delta y = 0 = \delta z,\ \delta t = \varepsilon x,$$

which can be written in Lie's fashion as

$$T_1 = t\frac{\partial\varphi}{\partial x} + x\frac{\partial\varphi}{\partial t}.$$

There are analogous transformations T_2 and T_3 found by replacing x with y and z respectively. Poincaré also defined the transformation found T_0 by setting $\varepsilon = 0$ and $l = 1 + \delta l$, when

$$\delta x = x\delta l,\ \delta y = y\delta l,\ \delta z = z\delta l,\ \delta t = t\delta l,$$

so

$$T_0 = x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z} + t\frac{\partial\varphi}{\partial t}.$$

These infinitesimal generators give rise to transformations that can all be written as composites of transformations of the form

$$x' = lx,\ y' = ly,\ z' = lz,\ t' = lt,$$

and those that preserve the quadratic form $x^2+y^2+z^2-t^2$. Poincaré now stipulated that these transformations form a group, in which l is a function of ε. The transformation that consists of a rotation of π around the y-axis forces equation (5) to imply that $l(\varepsilon) = l(-\varepsilon)$. But this transformation is its own inverse, and so $l = \frac{1}{l}$, and so $l = 1$.

In Sections 7 and 8 Poincaré showed that Lorentz's hypothesis was the only one compatible with the impossibility of detecting absolute motion, but he gave a different reason, true to lifelong emphasis on the idea of a group: the Lorentz transformations must form a group and therefore necessarily $l = 1$. This left the possibility of detecting absolute motion to those phenomena that were not of electro-magnetic origin, such as gravitation. In [18] Lorentz had argued that gravitation behaved under Lorentz transformations as the electro-magnetic forces do. To analyse this idea, Poincaré considered the effect of a Lorentz transformation on any function of time, position, and velocity such as would arise in any analysis of time, under the extra assumptions that any suitable law of attraction would reduce to Newton's law for bodies at rest, and would not disagree with astronomical observations of slowly moving objects. He found that if speeds faster than light are allowed, then time can pass negatively, and when he excluded this possibility he was left with the proposition that gravity would travel at the same speed as light, but he concluded that further investigations were called for.

Space and time

The most surprising consequence of this work is that it did not open the way for Poincaré to embrace space-time. In his opinion in 1912 [40, see *Dernières pensées* 108] what changed with the work of Lorentz was that there were now two principles that could serve to define space: the old one involving rigid bodies, and a new one to do with the transformations that do not alter our differential equations. The first one had been a fundamental epistemological principle for Poincaré, but the new one is an experimental truth. Poincaré had always insisted that geometry belonged to mathematics, and could not be subject to revision by experimenters, and so physical relativity must become a convention, but whereas our conventional knowledge of geometry had formerly been rooted in the group of Euclidean isometries, it could now be rooted in the Lorentz group. The Lorentz group would guarantee our equations, at the price of placing us in a four-dimensional space. In an unmistakable reference to the ideas of Minkowski, although he did not mention his name, Poincaré went on [40, *Dernières pensées*, p. 108]:

"Time itself must be profoundly modified. Here are two observers, the first linked to fixed axes, and the second to moving axes, but each believing themselves to be at rest. Not only any figure, which the first one considers as a sphere, appear to the second as an ellipsoid; but two events which the first will consider as simultaneous will not be so for the second. Everything happens as if time were a fourth dimension of space, and as if four-dimensional space resulting from the combination of ordinary space and of time could rotate not only around an axis of ordinary space in such a way that time were not altered, but around any axis whatever. For the comparison to be mathematically accurate, it would be necessary to assign purely imaginary values to this fourth coordinate of space. [...] the essential thing is to notice that in the new conception space and time are no longer two entirely distinct entities which can be considered separately, but two parts of the same whole, two parts which are so closely knit that they cannot be easily separated."

So, Poincaré concluded [40, *Dernières pensées*, p. 109]:

"What shall be our position in view of these new conceptions? Shall we be obliged to modify our conclusions? Certainly not; we had adopted a convention because it seemed convenient and we had said that nothing could constrain us to abandon it. Today some physicists want to adopt a new convention. It is not that they are constrained to do so; they consider this new convention more convenient; that is all. And those who are not of this opinion can legitimately retain the old one in order not to disturb their old habits. I believe, just between us, that this is what they shall do for a long time to come."

Poincaré chose to be a Galilean to the end.

'Conformal' maps of several variables

Another topic that occupied Poincaré for many years was the function theory of several complex variables, and he was instrumental in extending the methods of potential theory. By the time he published his paper [37] on the subject in 1907, the German mathematician Friedrich Hartogs had done important work on what domains in $\mathbb{C}^n$ can be the domains of holomorphic functions [10]. This inspired Poincaré to study the nature of maps between domains in two complex variables, and he was able to show conclusively that the boundaries of some domains are such that there can be no regular map between the interiors of these domains. This showed that the Riemann mapping theorem cannot be extended to two complex dimensions, but Poincaré gave no explicit examples, which were only exhibited for the first time in Reinhardt's [43] in 1921.

Poincaré first observed that in single variable complex function theory, one can ask for a map that takes a curve ℓ and a point m on ℓ to a curve $\mathcal{L}$ with a point M on $\mathcal{L}$. The map is required to map m to M and to be regular in a neighbourhood of m. Or, one can ask for a map taking a closed curve ℓ bounding a domain d to a closed curve $\mathcal{L}$ bounding a domain D. The first of these problems, which Poincaré called the local problem, is always solvable in infinitely many ways, and the second, 'extended', problem has a unique solution via the Dirichlet principle.

Poincaré now looked at the analogous problems for analytic functions of two complex variables, and found that they behave very differently. The local problem asks for a map of a three-dimensional 'surface' (a hypersurface) s with a point m to a three-dimensional hypersurface S with a point M that takes m to M and is regular in a neighbourhood of m. The extended problem asks for a map of a closed hypersurface s bounding a domain d to a closed hypersurface S bounding a domain D, and asks if there is a regular function that maps s to S and d to D.

Now, the extended problem in two complex dimensions always has a solution. Poincaré noted that this follows directly from one of Hartogs's theorems, and Poincaré also sketched his own proof of that result. However, as Poincaré showed, the local problem will not always have a solution because it asks for three functions that satisfy four differential equations. He deduced that the local question is one about types of surfaces, which can be classified according to their groups of analytic automorphisms (for, if two surfaces correspond under an analytic automorphism, their groups are necessarily conjugate, and so the surfaces belong to the same class). In particular, if a surface s admits only the identity analytic automorphism, then the local problem has at most one solution, else the automorphism can be used to generate a second solution. Poincaré relied on Lie's theory of transformation groups in Lie's *Theorie der Transformationsgruppen*, vol. 3, and Campbell's *Introductory Treatise* [4] to establish that there are 27 possible groups. He then showed explicitly that for most groups there is a hypersurface having that group as its analytic automorphism group, but some groups correspond to two-dimensional surfaces, so there are hypersurfaces that not analytically equivalent, and the main result of the paper is established. But Poincaré's account was very unspecific. He described the group of the hypersurface (hypersphere) with equation $z\bar{z} + z'\bar{z}' = 1$ explicitly (in Section 7), but all he did to exhibit a hypersurface with a different group was to indicate how its equation could be found by means of Lie's theory.

Conclusions

Poincaré thought deeply about what it is to do good mathematics, to find what he once called the 'soul of the fact' around which a body of theory grouped itself in the most perspicacious way and enabled to mathematician to do the most effective work. (A fuller account of Poincaré's work will appear in [8].) In both his pure mathematics and his work on mathematical physics he advocated the study of what he called form over matter, by which he meant the abstract system of relations rather than specific objects that obey those relations. He looked for analogies that would enable a system of relations to be moved, with a degree of fidelity, from one field to another. In all of these ways the group idea was vital to him. In number theory it enabled him to place the modular equation in a richer setting, one that opened the way for some types of Fuchsian functions to do arithmetic work. The group idea was crucial to his analysis of surfaces at the start of his career, and his work on 3-manifolds towards the end. It allowed him to classify domains of holomorphic functions in $\mathbb{C}^2$. It allowed him to give an account of how we construct space around us, and also of the space and time of contemporary physics (but not space-time, a step he refused to take). But in none of these areas did he then pause to study the groups in any detail. It was the idea of a group, not group theory, that animated him.

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Asymptotic series of Poincaré and Stieltjes

At the end of the nineteenth century both Henri Poincaré and the Dutch mathematician Thomas Stieltjes had their first publication on asymptotic series. The question arises whether they had contact with each other at that time on this new subject. Hasse van Boven, Rob Wesselink and Steven Wepster analyse both publications and investigate these simultaneous developments.

In 1886, two seemingly independent papers appeared that introduced the new subject of asymptotic series: one written by Henri Poincaré [1] in Paris appeared in the December issue of *Acta Mathematica*; the other consisted of the dissertation of the young Dutch mathematician Thomas Jan Stieltjes [6], produced under the supervision of Charles Hermite and defended in June, also in Paris.

At first sight, both texts have quite different goals. Stieltjes uses asymptotic series to find practical approximations for various functions and integrals, whereas Poincaré searches for formal, analytic properties of those series. Many modern textbooks on perturbation theory and asymptotic series mention both authors side by side (e.g., [9]). But how is it possible that two mathematicians introduce a new research area at nearly the same time and place? How similar are their ideas, and were they actually conceived independently? First, we analyse both papers; next we investigate possible forms of contact between the authors.

Divergent series

Poincaré discusses in his paper first a concrete example, i.e. Stirling's series development of the gamma function

$$\log\Gamma(x+1) = \tfrac{1}{2}\log(2\pi) + (x+\tfrac{1}{2})\log(x) - x + \frac{B_2}{1\cdot 2\cdot x} + \frac{B_4}{3\cdot 4\cdot x^3} + \frac{B_6}{5\cdot 6\cdot x^5} + \cdots,$$

where the B_k denote Bernoulli numbers. He soon directs his attention to the more general properties of asymptotic series, which he introduces as follows. Let $m_0 + \frac{m_1}{x} + \cdots$ be a not necessarily convergent series and denote by $S_n = S_n(x)$ the partial sum of the first $n+1$ terms. In Poincaré's terminology, such a series represents a function $F(x)$ asymptotically if

$$\lim_{x\to\infty} x^n(F(x) - S_n(x)) = 0 \quad \text{for fixed } n$$

and

$$\lim_{n\to\infty} x^n(F(x) - S_n(x)) = \infty \quad \text{for fixed } x.$$

The second condition excludes convergent series from being asymptotic representations.

In contrast, Stieltjes sets out first to discuss various difficulties encountered when dealing with divergent series; then he turns to the function $F(x) = m_0 + \frac{m_1}{x} + \frac{m_2}{x^2} + \cdots$ (where the series need not be convergent) subject to the condition that

$$\lim_{x\to\infty}\left(F(x) - \sum_{i=0}^{n-1}\frac{m_i}{x_i}\right)x^n = m_0.$$

He calls the series *semi-convergent* when the condition is fulfilled.

Poincaré's asymptotic representations and Stieltjes' semi-convergent series cover identical concepts which we now address as asymptotic series. Before considering the question whether they had a common source of inspiration, let us first see how these authors used these concepts in rather different ways.

Stieltjes' method

Stieltjes illustrates his method of approximating functions with a well-known example. He considers the logarithmic integral

Thomas Stieltjes

$$\mathrm{li}(a) = \int_0^a \frac{du}{\log u} = \lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon} \frac{du}{\log u} + \int_{1+\epsilon}^a \frac{du}{\log u}\right)$$

for $a > 1$. Using the substitution $u = e^{a(1-v)}$ and the first n terms of the geometric series expansion for $\frac{1}{1-v}$ he gets

$$\begin{aligned}\mathrm{li}(e^a) &= e^a \lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon} \frac{e^{-av}}{1-v}\,dv + \int_{1+\epsilon}^{\infty} \frac{e^{-av}}{1-v}\,dv\right)\\ &= e^a\left(\sum_{k=1}^{n} \frac{(k-1)!}{a^k} + R_n\right)\end{aligned}$$

where (putting $a = n+\eta$, with $0 \le \eta < 1$) the remainder term is

$$R_n = \lim_{\epsilon\to 0}\int_0^{1-\epsilon} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv + \int_{1+\epsilon}^{\infty} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv. \qquad (1)$$

Since he is dealing with divergent series, the problem is to find an optimal number of terms that gives the best possible approximation, i.e., the number n for which $|R_n|$ is minimal. Stieltjes evaluates (1) by putting $ve^{-v} = e^{-1-x^2}$ and $1 - v = t = a_1x + a_2x^2 + a_3x^3 + \cdots$; by way of differentiation and recurrent relations he finds the coefficients $a_1 = \sqrt{2}$, $a_2 = -\frac{2}{3}$, $a_3 = \frac{\sqrt{2}}{18}, \ldots$ Consequently,

$$-\frac{dv}{1-v} = \left(1 - \frac{\sqrt{2}}{3}x - \frac{1}{9}x^2 + \frac{\sqrt{2}}{270}x^3 + \cdots\right)\frac{dx}{x}$$

and also

$$e^{-\eta v} = e^{-\eta}e^{\eta t} = e^{-\eta}\left(1 + \eta t + \frac{\eta^2t^2}{2} + \cdots\right).$$

Combining these, he gets

$$\begin{aligned}&\int_{1-h}^{1-\epsilon} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv\\ &\quad = e^{-a}\int_{\epsilon\sqrt{\frac{1}{2}}}^{L} e^{-nx^2}\left(1 + A_1x + A_2x^2 + A_3x^3 + \cdots\right)\frac{dx}{x},\end{aligned}$$

where the A_j represent polynomials in η, L depends on h, and h is chosen so that the series in brackets is convergent on the integration interval. Attacking the second integral in (1) in the same way, Stieltjes finds

$$\begin{aligned}&\int_{1+\epsilon}^{1+k} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv\\ &\quad = -e^{-a}\int_{\epsilon\sqrt{\frac{1}{2}}}^{L} e^{-nx^2}\left(1 - A_1x + A_2x^2 - A_3x^3 + \cdots\right)\frac{dx}{x},\end{aligned}$$

where k is chosen so that the integral extends to the same L as before. Hence the constant term between brackets disappears in the sum of the two integrands; the division by x can be carried through and the limit $\epsilon \to 0$ can be taken; and finally the remaining parts $\int_L^\infty$ are argued to converge to 0 when $n \to \infty$. Termwise integration and working out of the polynomials A_j then gives him the remainder term as

$$\begin{aligned}R_n = e^{-a}\sqrt{\frac{2\pi}{n}}\biggl(&\eta - \frac{1}{3} + \left(\frac{\eta^3}{6} - \frac{\eta^2}{2} + \frac{\eta}{12} + \frac{1}{540}\right)\frac{1}{n}\\ &+ (\cdots)\frac{1}{n^2} + \cdots\biggr).\end{aligned}$$

Now, using $a = n+\eta$ and expressing η as a power series with unknown coefficients $\sum \frac{\beta_k}{n^k}$, he also reduces this expression for R_n to a power series in $\frac{1}{n}$; equating the coefficients of this series to zero and solving for the β_i leads finally to the identity

$$n = a - \frac{1}{3} - \frac{8}{405a} + \frac{16}{25515a^2} - \cdots \qquad (2)$$

for the supposedly optimal number of terms n in the divergent series approximation

$$\mathrm{li}(e^a) \approx e^a\left(\sum_{k=1}^{\lfloor n\rfloor} \frac{(k-1)!}{a^k} + \frac{\lfloor n\rfloor!}{a^{\lfloor n\rfloor+1}}(n - \lfloor n\rfloor)\right),$$

where $\lfloor n\rfloor$ denotes the largest integer not bigger than n; the last term is derived from a heuristic reasoning about R_n. In practice it is sufficient to compute (2) up to the quadratic term. Stieltjes adds that the 'order of approximation' is $\sqrt{2\pi/a}$, however, this is likely to be a very large estimate of the error (as he observes himself) and his concept of approximation order is not well defined. Moreover, Stieltjes shows a numerical example to demonstrate that his asymptotic series approximates the logarithmic integral much more rapidly than the classical result $\mathrm{li}(e^a) = \gamma + \log a + \sum_1^\infty \frac{a^n}{n\,n!}$.

Products of series

About a year after he had obtained his doctorate, Stieltjes proved that the product of asymptotic series is again asymptotic [7]. His proof is nearly identical to that which Poincaré had given in [1], except for a geometric consideration about the boundary terms. Poincaré's proof runs as follows. Let two asymptotic series be given:

$$J(x) = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \cdots,$$
$$J'(x) = A'_0 + \frac{A'_1}{x} + \frac{A'_2}{x^2} + \cdots + \frac{A'_n}{x^n} + \cdots,$$

and let S_n, S'_n denote their partial sums. Denote the product of the series by

$$\Sigma = B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \cdots + \frac{B_n}{x^n} + \cdots,$$

with partial sums Σ_n. Since S_n, S'_n and Σ_n are polynomials of degree n and $2n$ in $\frac{1}{x}$, we see that

$$\lim_{x\to\infty} x^n \left(S_n S'_n - \Sigma_n\right) = 0,$$

also

$$\lim_{x\to\infty} \frac{J}{S_n} = \lim_{x\to\infty} \frac{J'}{S'_n} = 1.$$

Finally, by definition we have

$$\lim_{x\to\infty} x^n \left(J - S_n\right) = \lim_{x\to\infty} x^n \left(J' - S'_n\right) = 0.$$

Writing $J = S_n + \frac{\epsilon}{x^n}$ and $J' = S'_n + \frac{\epsilon'}{x^n}$, we get

$$JJ' = S_n S'_n + \frac{S'_n\epsilon + S_n\epsilon' + \frac{\epsilon\epsilon'}{x^n}}{x^n}.$$

When $x \to \infty$, we have $S_n \to A_0$ and also $\epsilon \to 0$ and likewise for the primed magnitudes, hence

$$\lim_{x\to\infty} x^n \left(JJ' - S_n S'_n\right) = 0$$

and

$$\lim_{x\to\infty} x^n \left(JJ' - \Sigma_n\right) = 0,$$

i.e., JJ' is an asymptotic series.

This multiplicative property and its proof fit well within Poincaré's interest in their analytic properties. Stieltjes did not need such properties initially, when he limited himself primarily to concrete examples; it was only years later that he needed them. We remark that at that time he provided the proofs himself instead of referring to Poincaré's paper. Was he unaware of Poincaré's work?

Contact

We have no evidence that Poincaré and Stieltjes had been in contact prior to the publication of their initial work in asymptotic series. There is no known correspondence between them although we have many letters written to or by each individual to several other colleagues. It is not unthinkable that they met in person in Paris as they both resided there between mid 1885 and the end of the next year. However, such a meeting is likely to have left traces in letters, and we know nothing of the kind. Also, Stieltjes' own proof of the multiplicative property of asymptotic series may be an indication that he did not know of Poincaré's work: had he known it, he could simply have referred to Poincaré's proof of the same result. But on the other hand we have the similarity between their definitions of asymptotic series which indicates that there might have been at least indirect contact.

It is hard to imagine that they had not met. To wit, Poincaré had been a student of Stieltjes' supervisor Hermite at the École Polytechnique, and Stieltjes had a clear interest in Poincaré's work. On 19 March 1886 he wrote to his supervisor:

"Comme seconde Thèse, je voudrais bien exposer la démonstration due à M. Poincaré de la possibilité d'une figure annulaire d'une masse fluide en rotation. C'est un sujet qui m'intéresse beaucoup et j'ai encore quelques mois de temps, certainement, avant que ma Thèse ne soit imprimée." [8, Chapter I, p. 190]

Stieltjes already expressed his interest in Poincaré's work when he was still in Leiden, when the name of Poincaré was mentioned several times in the correspondence between him and Hermite. So Stieltjes certainly had Poincaré in view. Only years later did a reverse interest take shape.

Photo: Gauthier-Villars [8]

Charles Hermite, supervisor of Thomas Stieltjes

This asymmetry in their relationship may go some way to explain why (if so) they had not met in person. In fact, Stieltjes moved to Paris in the middle of 1885. At that time he did not yet have a reputation among mathematicians, although he had published a few papers and was already a member of the Royal Dutch Academy of Sciences (KNAW). Only when in Paris did his reputation rise. Meanwhile, Poincaré was an established mathematician (though still only 31 years of age) who had many contacts with colleagues, and the academically younger Stieltjes may have missed his attention.

External infuence

As already mentioned, Hermite might have provided the link between Stieltjes and Poincaré. He was one of the most important mathematicians then and counted many students, including Mittag-Leffler. After Stieltjes obtained his doctorate from Hermite and Darboux it was Hermite who helped him to acquire a position in Toulouse. Hermite and Stieltjes remained in friendly contact and they maintained an extensive correspondence in which more than 400 letters were exchanged.

Poincaré too kept in contact with his former teacher Hermite. Therefore Hermite was able to inform Poincaré that Stieltjes worked on the generalisation of certain ideas of Lagrange and Cauchy in complex function theory. Stieltjes abstained from publishing his results because he was unable to complete the proofs; Poincaré was more successful and published his results in [2], adding that he had elaborated on the initial ideas of Stieltjes.

This proves that Hermite *did* function as an intermediary between the two, at least sometimes. Since Poincaré's paper was published near the end of 1887, Hermite and Poincaré must have discussed this matter in that summer at the latest. Probably it was not the first time that Hermite passed on ideas to Poincaré.

Another link between Stieltjes and Poincaré might have been the editor of *Acta Mathematica*, Gösta Mittag-Leffler in Stockholm. Mittag-Leffler and Poincaré have maintained an extensive correspondence of which some 258 letters have been preserved [4]. In April of 1886, Poincaré wrote him:

"Je vous adresse aujourd'hui le mémoire que vous m'aviez demandé au sujet des intégrales irrégulières des équations linéaires et de leur représentation approximative par des séries divergentes analogues à celles de Stirling." [5]

His attached article discussed the properties of asymptotic series. He added that he had been unable to work on it for two months because of his health. This implies that he had already been working on the topic for a while.

Stieltjes also had contact with Mittag-Leffler, for he had published five papers in the *Acta* of which two dated from his Leiden years and three from France. They also exchanged four letters discussing Riemann's zeta-function (these have been published as an appendix to [8]). Clearly, Mittag-Leffler had less contact with the Dutchman than with Poincaré. Although the topic of asymptotic series is discussed among Mittag-Leffler and Poincaré, the former never mentioned Stieltjes' work on asymptotic series.

Conclusion

There is a notable similarity in Stieltjes' and Poincaré's work on asymptotic series, but we have found no proof of direct contact between the two. This is remarkable because they shared academic ancestry and friendship in the person of Charles Hermite, who had often advised both men. Hermite must have been aware of his students's projects yet it seems that he did not always advise them of common factors. Sometimes he did function as an intermediary as in the case of Stieltjes getting stuck on the residues of double integrals.

Years later, Poincaré reviewed an article of Stieltjes on continued fractions and (what we now know as) the Stieltjes integral (see [3]). Poincaré praised Stieltjes: "Le travail de Stieltjes est donc un des plus remarquables Mémoires d'Analyse, qui aient été ecrits dans ces dernieres années." It is impossible to tell whether he had formed this opinion recently or already around 1886.

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Poincaré and Brouwer on intuition and logic

In the beginning of the twentieth century the Dutch mathematician Luitzen Egbertus Jan Brouwer published his first papers on intuition and logic. There is no indication that Henri Poincaré was aware of these publications, but it would have been interesting to know what he had have to say about them. In this article Dirk van Dalen, Emeritus Professor Logic and Philosophy of Mathematics, compares the ideas of Poincaré and Brouwer on the foundations of mathematics.

The mathematical foundational landscape at the beginning of the twentieth century was dominated by late nineteenth century novelties, such as symbolic logic, set theory, and formalisation. The generally acknowledged grand master of the Foundations of Mathematics was Henri Poincaré. Not in the sense that he was himself involved in presenting novelties, but rather as a generally acknowledged universal creative mathematician, who could from the height of the Olympus survey, encourage and criticise the developments in the field. This did not mean that he did not actively study a specific more technical subject, but that he left his gifts for others to pursue. There is no doubt that in the first decades of the twentieth century he was the best and most widely read mathematical author. Whole generations of mathematicians were introduced into the intricacies of the foundations of mathematics by Poincaré's Flammarion books.

The purpose of this paper is to look at some specific issues in the œuvre of Poincaré and to compare them with the subsequent ideas of the newcomer L.E.J. Brouwer. As Gerhard Heinzmann and Philippe Nabonnand have already discussed most of the issues at hand in their magisterial paper 'Poincaré: intuitionism, intuition and convention' [14], the present paper can be seen as a footnote to it. I will restrict myself to a few topics that may be of interest.

The comparison of Poincaré and Brouwer will inevitably be somewhat out of focus, as Brouwer's mature foundational papers appeared only after Poincaré's death. There is no indication that Poincaré was familiar with Brouwer's early publications. In particular it is unlikely that he had seen Brouwer's dissertation, written in Dutch, which for a long time was the prime source of Brouwer's intuitionism. There was a contribution of Brouwer in the proceedings of the 1908 Rome conference, a conference that was attended by Poincaré. However, it would be difficult to get a balanced impression from such a condensed report. The surviving correspondence in 1911 between Poincaré and Brouwer deals with automorphic functions and uniformisation [10]. We may safely assume that Poincaré was not aware of Brouwer's 'other life'; hence it remains an open question what Poincaré would had have to say about this new actor on the stage of the foundations of mathematics.

Personal contact between Poincaré and Brouwer remained restricted to a few letters. Brouwer was an admirer of Poincaré, he highly valued Poincaré's work in topology and his contributions to the foundational debate around the century. Poincaré, who knew Brouwer as a topologist, appreciated the newcomer; his reply to a letter of Brouwer on the topic of automorphic functions closed with the sentence: "I am happy to have this opportunity to be in contact with a man of your merit."

It is clear that Brouwer was thoroughly familiar with most of Poincaré's papers; the dissertation contains a large number of references to that effect. A conspicuous exception can be found in the correspondence of Brouwer and Hadamard on 24 December 1909 [9], where Hadamard calls Brouwer's attention to Poincaré's paper on curves defined by differential equations (published in 1881).

On the whole the ideas of Poincaré and Brouwer show a strong similarity. The reader who consults their foundational publications will however be struck by a striking difference in style. Poincaré published for a large readership in mathematics and physics, and for the cultivated reader in general, as a result his style is literary and pedagogical; he had completely mastered the use of the turn of the century narrative scientific exposition. Brouwer, on the other hand, had no mercy on his readers; he shunned long explanations and indulged in archaic expressions.

About logic and logicians

Both Poincaré and Brouwer were critical of the role of logic in mathematics. There is however a marked difference in their views and reactions. In Poincaré's writings the work of Russell played a substantial role, Brouwer, on the other hand rejects Russell's approach of logic on the ground that logical principles hold only for words with a mathematical meaning, "and exactly because Russell's logic is nothing but a word system, without a presupposed mathematical system to which it applies, there is no reason why no contradictions should appear." It should not come as a surprise that Russell's monograph *An Essay on the Foundations of Geometry* is extensively discussed in Brouwer's dissertation — after all, it deals with mathematics. The last chapter of the dissertation contains a discussion of the various approaches to modern logic, and more or less turns them down on the grounds of his philosophical, constructive views. Poincaré, on the other hand follows Russell's logical theories with great attention. The difference in outlook between Poincaré and Brouwer is rather striking; Poincaré accepts logic as it is and seeks to safeguard it from the various dangers that had been discovered. In line with the contemporary literature, he attaches great value to the problem of predicativity. The pressing question here is: "Can one define a mathematical object using a class which contains that object?" This indeed is a traditional mathematical practice, used for example in the definition of supremum: the supremum s of a set A of reals is the least number of the class B of all numbers $\geq$ all numbers of A. Obviously, $s \in B$. Such a definition is called *impredicative*. In the logical literature of the beginning of the twentieth century the problem of predicativity plays a major role; the *vicious circle principle* explicitly forbids defining objects in terms of classes containing that object. Poincaré actively took part in analysing predicativity in the context of the Russell paradox, see *Les mathématiques et la logique, La logique de l'infini* [17]. On this issue Brouwer takes an independent position, according to him proofs are mental constructions, and (intuitionistic) logic has its own 'proof interpretation' (made precise by Heyting and Kolmogorov). Paradoxes of the Russell type thus ask for a proof construction that cannot be carried out. And thus no 'intuitionistic truth value' can be determined. Even in his later papers, where the so-called species are introduced, the predicativity issue is ignored (see also [12, p. 972; 13]).

Reading Poincaré's many accounts of, and objections to, logic, one gets the impression that he takes a rather 'physical' view of the subject. Just as in physical theories, there are external conditions that determine the applicability of logic. In *La Logique de l' Infini* Poincaré states for example that paradoxes arise because of the application of logic outside its proper domain, i.e. the universe where only sets with finitely many objects occur.

This statement occurs almost literally in Brouwer's *Intuitionistische Mengenlehre* [3, p. 2]: "In my opinion the Solvability Axiom [also known as 'Hilbert's Dogma'] and the principle of the excluded third are both false, and the belief in these dogmas historically is the result of the fact that one at first abstracted classical logic from the mathematics of subsets of a particular finite set, and next ascribed an a priori existence, independent from mathematics, and finally, on the basis of this alleged apriority, applied it to the mathematics of infinite sets." Brouwer, so to speak, traces the popularity of this dubious principle back to its historical origins. In his Berlin Lectures he offered again his interpretation of the long reign of the 'superstitious belief in the principle of the excluded third': "[It] can only be explained by the natural phenomenon, that many objects and mechanisms in the external world with respect to extensive complexes of facts and events can be controlled by considering and treating the system of states of these objects and mechanisms in the space-time world as part of a finite discrete system with finitely many connections between the elements of the underlying system, so that the principle of the excluded third turns out to be tangibly applicable to the relevant complexes of objects and mechanisms." [6, p. 22]

He was well aware of the fact that the principle of the excluded third could not simply be refuted by logic: "that nonetheless classical mathematics is not right away silenced, is due to the supporting circumstance that although the *principium tertii exclusi* is in fact incorrect, but, as long as one restricts its application to finite groups of properties, it is non-contradictory, so that intuitionism, when fighting the aberrations of classical mathematics, is deprived of the most widely accepted mode of repression of errors of thinking, the *reductio ad absurdum*, and has to rely exclusively on admonition to rational reflection." [4]

The above-mentioned principle of the excluded third (also called principle of the excluded middle, PEM) is the touchstone for the constructive nature of a theory. It states that any statement A is true or false, in symbols: $A \vee \neg A$ is true. On this Aristotelian principle the important and convenient *proof by contradiction* and *Consistency* ⇔ *Existence* are based. If one takes existence a bit more seriously, then "there is a solution for the equation $A(x) = 0$" means more than "it is impossible that there is no solution". One wants to produce the number a for which $A(a) = 0$ holds. In Brouwer's intuitionism *existence* is taken to mean *constructible*, therefore he had to revise logic. He did indeed formulate a constructive interpretation of logic, in particular of the hypothetical judgement [2, p. 125]. The above-mentioned 'proof interpretation', where proofs are mental constructions was the basis of a new and stricter logic.

Until the end of his career Brouwer stuck to his fundamental view on the role of logic: "Further there is a system of general rules called *logic*, enabling the subject to deduce from systems of word complexes conveying truths, other word complexes generally conveying truths as well. Causal behaviour of the subject (isolated as well as cooperative) is affected by logic. And again object individuals behave accordingly. This does not mean that the additional word complexes in question convey truths *before* these truths have been experienced, nor that these truths *always can* be experienced. In other words, logic is not a reliable instrument to discover truths and cannot deduce truths which would not be accessible in another way as well." [5] In short: "There are no non-experienced truths and [that] logic is not an absolutely reliable instrument to discover truths."

We see that both Poincaré and Brouwer had their reservations about logic. But their motivation was totally different. In the wake of the paradoxes of Richard and Russell Poincaré saw the problems in logic as technical issues in second- or higher-order logic, shortcomings that could be corrected. Brouwer's scepticism concerned logic *tout court*, already propositional logic was suspect. Consequently Poincaré did not revolutionise logic, he suggested various medicines for the patient, whereas Brouwer completely revised logic on the basis of his thesis "a proof of a statement is a construction". The first steps were taken by Brouwer in his dissertation, where he formulated the underlying idea of the proof interpretation (see [1, 8]). The radical revision of logic paid off in due time, but these first steps required a young radical and not an elderly statesman. Indeed, the failure to deal with the non-effective aspects of logic left the French semi-intuitionistic with a half-hearted program.

In discussions of semi-intuitionism there is always a certain believe or hope that here is the place where constructivism was born. This does not seem justifiable; although certain distinctions were discussed, a wholesale overhaul of mathematics was impossible without a revision of logic. In Poincaré's case a rejection of the constructive tenets is embodied in his slogan: "What does the word existence mean in mathematics? It means freedom of contradiction." (*Les derniers efforts des logisticiens*, [16].) Indeed, it would be hard to imagine a conversion of the prolific Poincaré to a frugal mathematical world of constructivism, but he might very well have recognised Brouwer's mathematics as a viable alternative to the traditional one.

Nonetheless it would have been most illuminating to see his reactions to Brouwer's program; as Couturat could testify, Poincaré was not used to mince words.

Intuition

Poincaré may not have been the first mathematician of the new generation of the end of the century to advocate the restoration of intuition to its legitimate position, but he certainly was the most persistent one. His popular expositions ring with praise of the role of intuition in mathematics, contrasting it in particular with the clerical virtues of logic.

Mathematicians would read his version of intuition mainly as the human capacity to make in mathematical research choices based on an assortment of insights and experiences acquired by the subject. It is indeed this aspect that is highly valued by Poincaré, and probably by almost every mathematician, but there is also the other notion of intuition, called *Anschauung* by Kant. The latter notion is duly discussed by Poincaré and the role of non-Euclidean geometry is discussed in detail, but there it more or less stops.

Moving to Brouwer, we note that in a bold move he posits the so-called *ur-intuition* as the unique basis for mathematics. In one stroke the subject introduces both the discrete (natural) numbers and the continuum, see the rejected parts of Brouwer's dissertation [7, 18]. In later publications the expression 'move in time' is introduced to elucidate the time/continuum intuition. The characteristic of the continuum is expressed in the dissertation as: "Recognising the continuum intuition, the 'flowing', therefore as primitive, as well as the joining in thought of various things as one, which is the basis of any mathematical structure, we can name properties of the continuum as 'matrix of all points' that

Luitzen Egbertus Jan Brouwer (1881–1966) can be thought as a whole." Since the mathematical continuum is identical with the time continuum, it is interesting that in Brouwer's notes for the dissertation the creation of the time as matrix of moments, is called a free act of ourselves, and "with that creation at the same time the conditions and all elements for the construction of the whole of mathematics are given; one of these is the three-dimensional Euclidean geometry, and that is a suitable schema to manage in a simple language a group of phenomena, ..."

Here Brouwer and Poincaré share a vision of the continuum as an amorphous, immediately given medium (see [15]). With Poincaré this is an interesting comment on the intuitive character of the continuum, but no further analysis is made. Brouwer did go further by making this intuitive continuum the cornerstone of his mathematics (together with the natural numbers). In the dissertation he turned the intuitive continuum into a measurable continuum which made it amenable to the standard mathematical practice. After his introduction of choice sequences, based on free will in 1918, he could furthermore establish a number of basic properties of the amorphous continuum. For Poincaré these results would have been out of bounds as they were in direct conflict with classical logic.

In spite of the obvious parallels between Poincaré's and Brouwer's foundational views, here Poincaré's and Brouwer's paths separated. All this must be stated with a serious proviso: Poincaré never saw Brouwer's new mathematical universe. He may well have strongly objected to the subjective element in intuitionism had he lived longer. But it is equally possible that with his strong intuitions, he would have recognised the viability and legitimacy of choice objects in a revised logical setting.

On the issue of choice elements mathematicians had been very cautious. Non-law-like sequences occur presumably for the first time with Paul DuBois-Reymond [11], they next occur with Borel. Whereas DuBois-Reymond hardly elaborates the underlying ideas, Borel discusses choice sequences in a number of publications. His ultimate conclusion is that the notion is interesting, but does not belong to mathematics proper. In itself this is not surprising, as a convincing treatment of choice objects demands a constructive logic. Hence that road was closed to Borel, and presumably also to Poincaré.

Almost all mathematicians will agree that the castle of mathematics could not be built on a foundation without natural numbers. On this point Poincaré and Brouwer are in full agreement. Their writings show us similar reflections on the topic. The catchwords here are *iteration* and *induction*. If there is any distinction at all, it is that with Poincaré mathematical induction is a prime notion. At various places he proclaims the principle of mathematical induction as 'a truly synthetic *a priori* judgement'. On the other hand at just as many places he presents iteration as directly given by intuition. He indeed falls back on iteration (or recurrence) to motivate (or prove) mathematical induction. The argument is as natural as it is simple: let $A(n) \to A(n+1)$ be true for all n and let $A(1)$ be true, then, since $A(1) \to A(2)$ is true, also $A(2)$ is true. Now from $A(2)$ is true and $A(2) \to A(3)$ is true, it follows that $A(3)$ is true. By iteration, that is repeating the same operation, one gets that $A(n)$ is true for each n.

A similar effect can be seen in Brouwer's approach, the difference being that Brouwer accepts *iteration* as immediately given by intuition. In later publications this act is described as the self-unfolding performed by the subject, and immediately provided by intuition. Induction thus becomes a consequence of iteration. In *Science et Hypothèse* Poincaré explicitly expresses the same view: "The power of the mind which knows itself capable of conceiving the unlimited repetition of the same act once this act is possible. The mind has a direct intuition of this power." We may thus claim that Brouwer and Poincaré were in complete agreement on the role of iteration and induction. Since Poincaré's *Sur la nature du raisonnement mathématique* goes back to 1894, and was re-issued in *Science et Hypothèse* (1902), it is not unreasonable to guess that Brouwer may have been influenced by Poincaré.

Methodological reflections

The discovery of non-Euclidean geometry, and the interrelations between the various geometries heralded the downfall of the doctrine that our knowledge of space is a priori. And thus the choice of geometry, for example for physical theories, became a matter of convention. Poincaré elaborated the philosophy/methodology of the resulting conventionalism in a large number of publications, for a precise analysis see [14]. To quote just one characteristic statement of Poincaré on the topic: "Next must be examined the frames in which nature seems enclosed and which are called time and space. [...] it is not nature which imposes them upon us, it is we who impose them on nature because we find them convenient." [15]

Brouwer's methodology for connecting (parts of) the outer world and suitable theories is based on a different ideology, but results in something rather similar. There are few places where he dwells on this issue, e.g. the dissertation. In the chapter 'Mathematics and Experience' Brouwer explains the unexpected success of mathematics in dealing with the natural world. From the intuitionist point of view the outer world consists of the sensations of the subject modulo abstraction under similarity (the technical term is *causal sequence*), i.e. sensations that are similar from a particular point of view are identified, thus yielding *objects*. The resulting system of objects and their relations is then further abstracted to a mathematical system. These mathematical systems are purely abstract conglomerates based on the ur-intuition; they are waiting to be applied. The choice of the mathematical system is up to the subject; he can extend the system to a wider one, which is often useful in simplifying parts of the old one, and which opens up the possibility of 'prediction'. The subject is free however to revise such extensions, should they conflict with the causal sequences in the outer world (be refuted by experiments). Without going further into Brouwer's theory of science, which is cloaked in terms of the mental activity of the subject, we see that the relation physics–mathematics (Poincaré) matches the relation outer world–mathematics (Brouwer).

Brouwer's outer world, which consists for the subject in highly stable or invariant causal sequences (equivalence classes of similar sensations) is after all not that far from Poincaré's 'objective reality': "But what we call objective reality is, in the last analysis, what is common to many thinking beings, and could be common to all; this common part, we shall see, can only be the harmony expressed by mathematical laws."

The two champions of intuition

Comparing the two grandmasters of topology and the philosophy of mathematics, one is struck by the differences in presentation and in philosophical position. Poincaré's writings belong to the era of the literary giants of the nineteenth and early twentieth century; he addresses the educated layman as well as the specialist, and cultivates a wonderfully balanced style. The essays of Poincaré on an immense variety of foundational topics almost invariably start from an elementary level, and move up with a wealth of subtle arguments and examples to the issues of the day.

He elaborates most of the issues of the exact sciences and at the same time stresses the ethical and moral aspects that are usually relegated to their place 'between the lines'. The introduction to *La Valeur de La Science* opens with the memorable: "The search for truth should be the goal of our activities; it is the sole end worthy of them." And after that the warns that the search of truth demands utter independence from the individual, whereas we usually derive strength from being united with others: "This is why many of us fear truth; we consider it a cause of weakness. Yet truth should not be feared, for it alone is beautiful." Few expositions of science contain such exhortations — for that reason alone reading Poincaré should be obligatory for students.

His mathematics is also presented in the admirable discourse of the nineteenth century intellectual. The immense popularity of Poincaré's Flammarion books testifies to his considerable gifts as an educator.

There is a striking contrast with Brouwer's policy and style. There is no doubt that Brouwer was equally sincere in his wish to improve, or even save, the world. But where Poincaré cultivated the role of a wise but stern teacher, who knew well that the reader is sooner convinced by an instructive and pleasant discourse, than by a grim sermon presented by an inflexible preacher, Brouwer had no compassion with his audience or readership. Compared to Poincaré he was an old testament prophet who predicted the end of the world, unless ... His admonitions in *Life, Art, and Mysticism* were harsh and uncompromising. The influence of this little read monograph was negligible, but that did not stop Brouwer's efforts to convert the mathematical community with well-chosen and refined attacks on, in his eyes, foolish convictions. The Vienna lecture 'Mathematik, Wissenschaft und Sprache', which was, by the way, Brouwer's first exposition of his philosophy to appear in print, may serve as an example.

Even his mathematical publications were held in awe because of their merciless exactness and parsimony with elucidation; nobody less than Hausdorff complained that "The brevity of Brouwer's papers, which often forces the reader to fill in many details by himself, is most regrettable, in the absence of other impeccable and extensive expositions." The modern reader, however, will be pleasantly surprised with this Bourbaki *avant la lettre* directness of exposition.

Conclusion

Summing up, many of the issues in Poincaré's leisurely expositions, of a strongly methodological nature, reappear in Brouwer's work, be it in a concise and precise way. The decisive step made by Brouwer beyond Poincaré's contributions was his abandoning Aristotelian logic and his switch to a rigorous constructive position, based on the intuition of the subject (including choice sequences). Thus raising the level of the discourse to a higher exactness and precision.

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Poincaré and Analysis Situs, the beginning of algebraic topology

In 1895 Henri Poincaré published his topological work 'Analysis Situs'. A new subdiscipline in mathematics was born. Analysis Situs was an inspiration to new fields like algebraic topology, Morse theory and cobordism. With use of today's knowledge and notation, Dirk Siersma views back to this historical work of Poincaré.

What was the impact of Poincaré on topology? He introduced the concept of manifold in any dimension and defined homologies and fundamental groups. This was the starting point for the development of algebraic topology. Although he discusses the general case, his work is quite concrete. He works often with examples and makes computations. This was his way to get intuition. His topological work 'Analysis Situs' [5] appeared in 1895. Before, in 1892 he had published a short (four pages) announcement in *Comptes Rendus* [4].

Analysis Situs describes the relative position between objects (points, lines, surfaces) without bothering about their sizes.

Analysis Situs is written in an intuitive style, which is quite different from the present mathematical writing. It reads sometimes like a novel. It is divided into 18 short chapters and consists of 121 pages. Definitions and theorems are not so often mentioned as such. Poincaré is not always precise and at some places there are gaps and mistakes. Due to criticism of other researchers (e.g. Heegard) he responded by adding supplements (all together five) during the period 1899–1904. In the last (fifth) supplement he stated correctly his question, which we call now the Poincaré conjecture (which was proved by Perelman in 2003).

Analysis Situs and the supplements contain (in a preliminary stage) many seeds for further developments: algebraic topology, Morse theory, topology of algebraic varieties and cobordism. This article is not a historical survey of Poincaré's topological works. It reports on my experiences while reading in his work. At several places I will be anachronistic and use some of today's knowledge and notations and view back to Poincaré's work.

Several books have been written about Poincaré's topological work. We mention first John Stillwell's English translation [6] of Analysis Situs and the five supplements, which appeared under the title *Papers on Topology* [6]. As further reading I propose the article of Sakaria [7] and the book of Scholz [8].

Why Analysis Situs?

How did Poincaré come to study analysis situs? Most of his work was of a geometric nature: differential equations (in his dissertation), dynamical systems and the theory of automorphic functions. This last subject is related to non-Euclidean geometry. In his study of differential equations he was also looking to more qualitative aspects. e.g. the indices of zero's of a vector field and more global aspects of the theory. An example is the index formula for vector fields: the sum of the indices on a surface of genus p is equal to $2-2p$. So it only depends on the 'shape' of the surface. Moreover he wanted to generalize this to higher dimensions. He saw a need for extending the concept of connectivity (in the surface case related to the genus) to higher dimensions.

He also looked at spaces of differential equations on algebraic curves. Depending on genus and branching order he constructed an object, depending on many coordinates, which he called *multiplicité*. In his theory of automorphic functions he found in a similar way a *multiplicité* of Fuchsian groups with fundamental region a surface of genus p and given branching. Also in his work on double integrals on $\mathbb{C}^2$ he entered the theory of submanifolds of $\mathbb{R}^4$. At several places he talks about the need of a 'hypergeometrical language'. In 1892 [4] it was so far that he announced Analysis Situs as new subdiscipline in mathematics.

Manifolds

Before Poincaré the concept of (smooth) manifold was already used in the two dimensional case: classification of embedded surfaces in $\mathbb{R}^3$ was carried out by Möbius in 1863. There was also a description by identification and by fundamental region. The notion of an n-dimensional manifold was already around and used by e.g. Betti.

Poincaré does not give an abstract definition of a manifold, but describes them by constructions. See also Figure 1.

The first construction was by a set of p *equations* in $\mathbb{R}^{n+p}$ with Jacobian matrix of maximal rank together with some inequalities. This is nowadays called the submersion condition.

With the second construction he could describe more complicated situations: by *local parametrizations*; in modern language a local embedding $\mathbb{R}^m \to \mathbb{R}^n$. Poincaré relates the first construction to the second by the implicit function theorem. He also discusses the overlap between several local parametrizations, as a chain (like in the case of analytic continuation of complex functions) but without the concept of atlas.

In chapter 10 he considered a third construction *Geometric Representation*, where a certain number (one or more) of polyhedra in ordinary space are glued together by identifying pairs of faces. (Of course the gluing has to be done in such a way that the result is a manifold!)

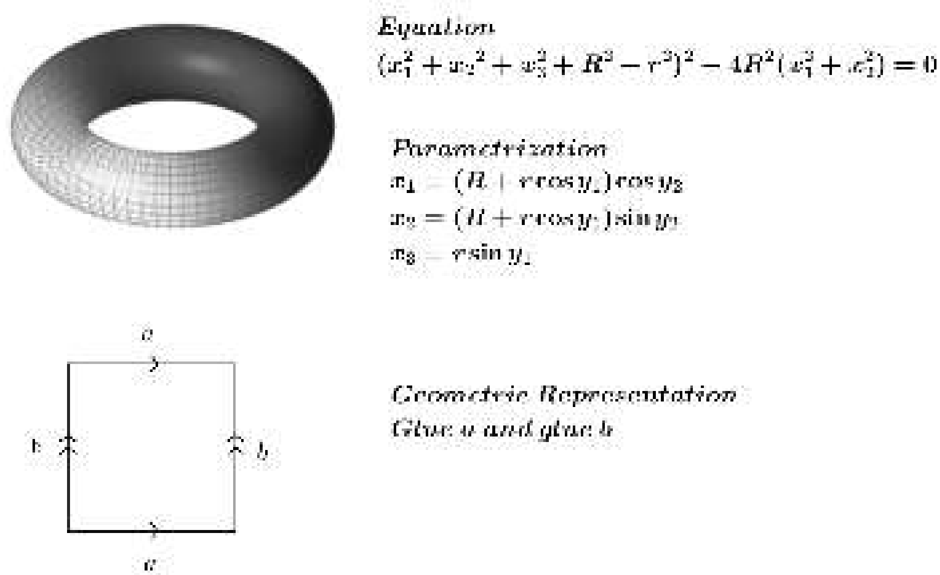

Figure 1 Three types of definition of a manifold

Main examples are cube manifolds, which we will discuss later. Anyhow in the geometric representation Poincaré made most of his computations.

The definitions also allowed manifolds with boundaries, e.g. the solid torus, the n-ball and the regions between two spheres.

Homologies and Betti numbers

Poincaré wanted to study the (higher) connectedness of a manifold. For this he introduced a calculus with submanifolds. He wrote:

$$k_1V_1 + k_2V_2 \sim k_3V_3 + k_4V_4, \quad k_i \in \mathbb{N},$$

when there exists a submanifold W with a boundary, which is composed of k_i copies of closed submanifolds V_i (Figure 2 and 3). He said: "Relations of these forms are called homologies." And moreover: "Homologies can be combined like ordinary equations." He defined the submanifolds $V_1, \ldots, V_\lambda$ to be independent if they are not connected by any homology with integral coefficients. He defined the connectivity of V with respect to manifolds of dimension m as P_m, if there exist $P_m - 1$ closed submanifolds of dimension m, which are linearly independent, but not less. So we get a set of numbers $P_1, \ldots, P_n$ for each manifold V of dimension n. He called this the sequence of *Betti numbers*. Note that these Betti numbers are 1 higher than today's Betti numbers (which are the ranks of homology groups). The Betti numbers occur also in the last chapter, where he generalized the Euler formula for surfaces to manifolds.

To allow negative coefficients he used the concept of orientation (Klein, van Dyck) in relation with the sign of Jacobian determinant of transition maps in the second construction of manifolds. This allowed him to write:

$$k_1V_1 + k_2V_2 + \cdots + k_\lambda V_\lambda \sim 0, \quad k_i \in \mathbb{Z}.$$

Although this is a linear combination of submanifolds, Poincaré did not consider the group theoretic aspects. He exploited the idea of Betti to consider 'taking-the-boundary' in order to measure connectivity. This became the main tool in geometric homology theories and cobordism.

In Analysis Situs he did not consider torsion. Poincaré allowed divisions: $4v_1 \sim 0$ implies $v_1 \sim 0$. In modern language he worked only with the free part of the homology groups. He discussed torsion in the first supplement (after criticism of Heegaard).

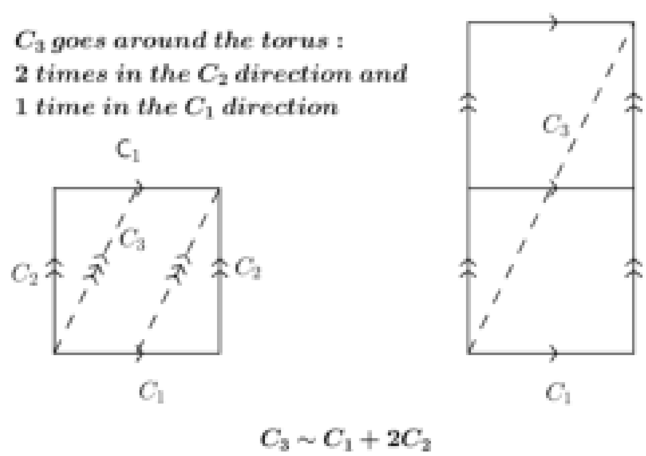

Figure 2 Homology on the torus

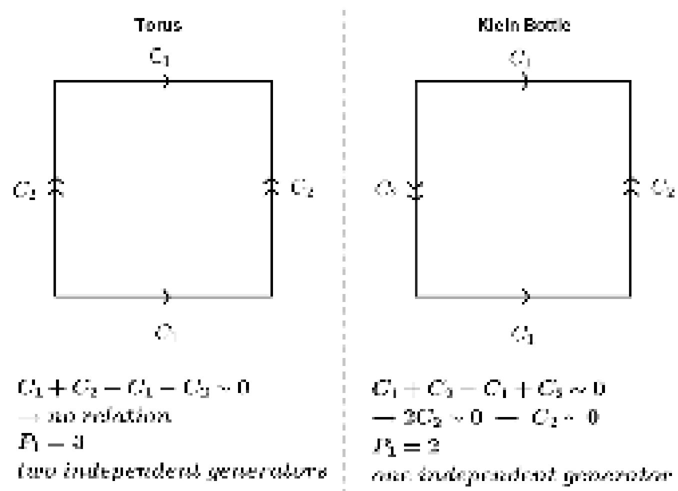

Figure 3 Betti numbers and homologies

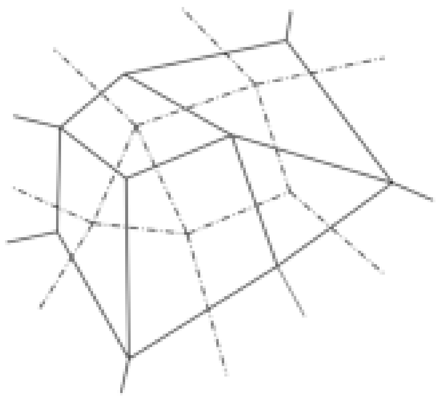

Figure 4 Cell decomposition and its dual

Poincaré duality

In examples it turned out that Betti numbers were symmetric around $\frac{n}{2}$. This is the so-called Poincaré duality, which is valid for oriented closed manifolds. He gave in chapter 9 a sketch of the proof of $P_k = P_{n-k}$. The central idea is to consider the intersection number of two (transversal) submanifolds of complementary dimension. For each intersection point this is the local intersection number $+1$ or -1, according to orientation of a system of tangent vectors. He defined the global intersection $N(V, V')$ as the sum of the local intersection numbers. He also claimed the independence under homology relations. It follows that for every $n-k$ cycle C there exists a closed k-dimensional submanifold V such that $N(V, C) \neq 0$. This explained the duality (anyhow for the free part of homology).

The criticism of Heegard (who showed him a counter example) was reason for him to describe in more detail the difference between homology with division and homology without division (including torsion). This was also a reason to produce a new proof for the duality (in the first supplement), where he looked to a decomposition into cells (homeomorphic to the ball) together with a dual cell decomposition (Figure 4).

The fundamental group

In chapter 12 Poincaré introduced the fundamental group. He knew from the theory of Fuchsian groups already the relation between closed curves on a surface and the substitutions in a system of multivalued functions. In the case of a 2-torus one can e.g. consider the two angular coordinates (which are defined local). See Figure 5. In fact if ϕ is such a coordinate its differential $d\phi$ is well defined and it gives rise to a multivalued function on the torus. The integral of $d\phi$ over a closed contour gives a integer multiple of 2π. The use of substitutions is quite typical for the period 1880–1920. It occurred also in systems of solutions of differential equations, following these solutions around different loops around singularities.

In general a contour produces a substitution in a multivalued function and a composition of contours results in a composition of substitutions. Multivalued functions can be interpreted as univalued on a certain covering space of the manifold. Substitutions act as deck transformations. In fact the 'group of substitutions' is a holomorphic image of the fundamental group. Be aware that no abstract concept of group was known. A 'group' was always connected with an action.

Poincaré's composition of closed curves (contours) with common base point is not commutative, but he used an additive notation. A first definition of equivalent contours (written as $\equiv$) was close to the homology relation. This definition was not completely clear and some corollaries were incorrect. He comes back to it in the fifth supplement, where he used continuous homotopy between contours in the modern sense of the term.

He also made the difference clear between homology ($\sim$) and homotopy ($\equiv$). In homotopy:

- composition is not commutative
- all contours have the same base point
- $nA \equiv 0$ not necessarily implies $A \equiv 0$ (note that Poincaré did not consider torsion in homology in 1895)

A next step (in chapter 13) was to describe the fundamental group by generators and relations. Generators are a finite number of principal substitutions $S_1, \ldots, S_p$ that correspond to closed contours $C_1, \ldots, C_p$ such that any other contour is equivalent to a combination of these fundamental contours in a certain order. These fundamental contours are not, in general, independent, and there are certain relations between them which are called *fundamental equivalences*. The fundamental equivalences enable us to know the structure of the group.

Figure 5 Angular coordinate as a multivalued function

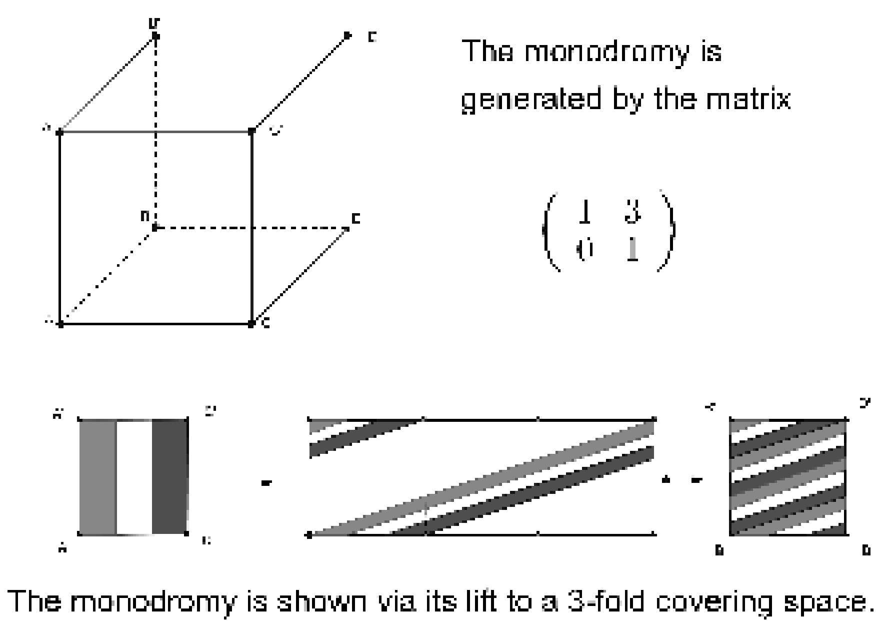

Figure 6 The construction of a cube manifold

Examples: The cube manifolds

Poincaré studied the cube manifolds as an important set of three-dimensional examples.

Consider the manifold V as orbit space of a group generated by the following three transformations:

$$g_i : \mathbb{R}^3 \to \mathbb{R}^3 \quad (i = 1, 2, 3)$$

defined by

$$g_1(x, y, z) = (x + 1, y, z),$$

$$g_2(x, y, z) = (x, y + 1, z),$$

$$g_3(x, y, z) = (ax + by, cx + dy, z + 1),$$

where a, b, c, d are integers and $ad - bc = 1$.

The fundamental domain is a unit cube. One identifies opposite faces by the following maps: the identity for the x and y coordinates and in the z direction a diffeomorphism, which is generated by a linear map (Figure 6).

Vertical sections of the cube correspond to tori. We can consider the cube manifold as a torus bundle over a circle. Such a bundle has a so-called *monodromy*. Cut the circle and look to the induced (trivial) bundle over the interval. The monodromy is the gluing map of the tori above the two end points. For cube manifolds the monodromy is generated by the linear map.

Figure 7 A handle body of genus 2

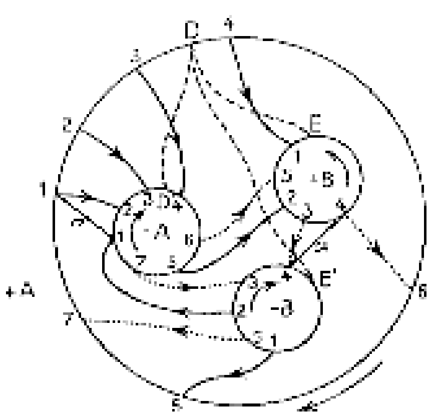

Figure 8 The Poincaré sphere

Poincaré sphere: explicit computation

Start with a 3-ball. On its boundary S^2 one chooses two pairs of discs $(+A, -A)$ and $(+B, -B)$. Glue $+A$ (which is shown as the outside region in Figure 8) with $-A$ and glue $+B$ with $-B$. Call the new boundary contours δA and δB. The resulting 3-ball with two handles is the handle body V_1. Its boundary W is a surface of genus 2.

Consider the contours:

$$C_1 = \delta A,\ C_2 = [\text{connects +A and }-\text{A}],\ C_3 = \delta B,\ C_4 = [\text{connects +B and }-\text{B}].$$

These four cycles are the fundamental 1-cycles of W, the fundamental group is free and abelian and equal to the first homology group of W. The fundamental group of V_1 is also free and abelian, generated by C_2 and C_4 and equal to the first homology group of V_1 (C_1 and C_3 are principal cycles).

Consider next V_2, another 3-ball with two handles. Glue now the two handle bodies together such that the principal cycles of V_1 are mapped to the cycles given by the unbroken and dotted lines in the picture. In terms of the generators:

$$3C_2 + C_1 + C_2 - C_3 + C_2 - C_4 - C_3 + 2C_4,$$

$$-2C_4 + C_3 - C_2 - C_4 - C_3 + 2C_4 - C_2$$

(additive notation, not commutative in the fundamental group).

First on the level of homologies he reduces by a detailed computation to two generators with two relations:

$$3C_2 + 2C_4 \sim 0,$$

$$-C_4 - 2C_2 \sim 0.$$

This set of linear equations has determinant -1 and so the first Betti number is 1 and there is no torsion. This space has the same homology invariants as the 3-sphere.

Next he shows (again explicit) that the fundamental group is non-zero; generated by C_2 and C_4 with relations

$$-C_2 + C_4 - C_2 + C_4 \equiv 0,\ \ 5C_2 \equiv 0,\ \ 3C_4 \equiv 0.$$

This is the *icosahedral group*. It's commutative image (the first homology group) is trivial. So we have a homology 3-sphere with non-trivial fundamental group!

Poincaré used the cube model to give a presentation of the fundamental group. He started with the 1-skeleton of the cube and added relations according to the two-dimensional faces. Next he computed the Betti numbers (P_1 by abelinization and P_2 by duality):

$$P_1 = P_2 = 2 \text{ in case } (a-1)(d-1) - bc \neq 0,$$

$$P_1 = P_2 = 4 \text{ in case } a = d = 1,\ b = c = 0,$$

$$P_1 = P_2 = 3 \text{ in other cases.}$$

Finally he looked for conditions when two of these fundamental groups are isomorphic. A necessary condition is the conjugation of the two groups. He concluded that there are infinitely many different manifolds with the same Betti numbers.

The Euler–Poincaré characteristic

Euler already showed the formula $V - E + F = 2$ for the number of vertices V, edges E and faces F of a convex polyhedron in $\mathbb{R}^3$. Poincaré generalized this in chapters 16–18 to arbitrary closed manifolds of any dimension p. Given a decomposition in polyhedral cells he looked at the alternating sum of the number of cells of dimension i (denoted by α_i):

$$N = \alpha_p - \alpha_{p-1} \cdots + (-1)^p \alpha_0.$$

Next he showed that N does not change under subdivision. Assuming that polyhedral decompositions always exist and that it is pos-

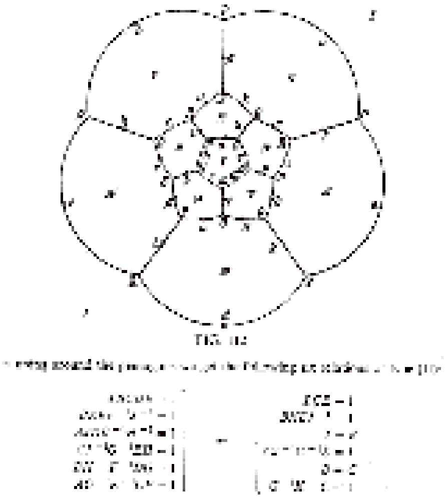

Figure 9 Spherical dodacahedron space

sible to pass to another by a sequence of subdivisions one gets that N does not depend on the polyhedral decomposition. Moreover he showed that N only depends on the Betti numbers of the manifold, so is in fact a homology invariant. In modern language we call the number N the *Euler–Poincaré characteristic* χ. By duality $\chi = 0$ for odd-dimensional manifolds.

The Poincaré sphere

Poincaré asked in [4] the question: "Can two manifolds have the same Betti numbers, but different fundamental groups?" An interesting test case for this is of course the 3-sphere. In Supplement 2 (where he was already aware of torsion) he even "confined himself by stating the following theorem the proof of which will require further developments":

Each polyhedron, which has all its Betti numbers equal to the Betti numbers of S^3 and has no torsion is homeomorphic to S^3.

Later on in Supplement 5 he disproved this statement via a manifold, which we call now the Poincaré sphere.

He constructed this manifold as follows: Consider two three-dimensional manifolds, in fact handle bodies, with the same surface as boundary and next glue these two together by a diffeomorphism of the boundary.

Note that given any 3-manifold, there exist always a splitting into two such handle bodies: a Heegard decomposition. Poincaré studied these handle bodies (see Figure 7) in detail and showed, that on each handle body there exists a system of so-called principal cycles. For computation of the fundamental group (and homology) of the handle body one can start with the presentation of the boundary surface and add these principal cycles as extra relations.

From a Heegard decomposition with separating surface of genus two Poincaré constructed his homology 3-sphere. See Figure 8. By duality we only have to look to the fundamental group and 1-homologies. He showed (see the explicit computation) that the fundamental group is the *icosahedral group*. Its commutative image (the first homology group) is trivial. So we have a homology 3-sphere with non-trivial fundamental group!

Next he stated his question: "Is it possible to have a 3-manifold with trivial fundamental group which is not diffeomorphic to the 3-sphere." This became the famous Poincaré conjecture (proved by Perelmann in 2003).

It is nowadays more common to describe the Poincaré sphere by conjugating facets of a regular dodecahedron. This space arises by identifications opposite face with a twist of $\frac{\pi}{5}$. This was checked by Kneser [2] in 1929. See Figure 9 taken from [9]; for details see its page 224.

The Poincaré sphere appears also in the theory of singular hypersurfaces in $\mathbb{C}^3$. It is the intersection of a small 5-sphere around the origin with this singular complex surface $x^2 + y^3 + z^5 = 0$. This type of intersection is called a link of a singularity. Links of $x^p + y^q + z^r = 0$ (p, q, and r pairwise relatively prime positive integers) are known to be homology spheres (named Brieskorn spheres).

Conclusion

In this article we could only describe a restricted part of Analysis Situs and its supplements. There are many aspects left, which I have not touched. There is a discussion about the triangulability of manifolds in Analysis Situs, which is continued in the supplements. It took until 1934 that Cairns proved in full rigour the statement that every differentiable manifold has a polyhedral subdivision. In supplement 3, 4 and 5 there is a description of the topology of algebraic surfaces in $\mathbb{C}^3$. These are four-dimensional spaces. Here one already can see the concepts of monodromy and vanishing cycles, related to complex Morse theory. This seems to be the beginning of singularity theory. Another new subject is the dynamic approach towards an evolution of a manifold from a simpler one (e.g. the n-ball), by attaching a series of handles. This is a beginning of Morse Theory.

Only a small part of Poincaré's work was devoted to topology. With Analysis Situs he started this new subdicipline in mathematics. Successors of Poincaré were mostly outside France. We mention Brouwer (in Holland), Heegard (in Denmark), Dehn en Hopf (in Germany). The first textbooks appeared in 1930 (*Topology* by Letschetz [3]) and in 1934 (*Lehrbuch der Topologie* by Seiffert and Threllfall [9]). As Lefschetz wrote later: "Perhaps no branch of mathematics did Poincaré lay his stamp more indelibly than on topology." I refer to the book of Dieudonné [1] for the continuation of this field. There was a lot of work left to make the theory completely rigorous; but there were plenty of idea's available for future developments.

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Perspectives on the legacy of Poincaré in the field of dynamical systems

Henri Poincaré (1854–1912) has had a tremendous influence on the development of mathematics and mathematical physics, the traces of which can be seen till the present day. This paper by Henk Broer attempts to highlight aspects of this legacy and its influence on certain more recent developments, focusing on the field of dynamical systems. His various approaches have invariably led to fruitful subfields, where geometry, algebra and later also measure theory are effectively at work in combination with analysis and calculus.

Poincaré lived in the happy days when the boundaries between mathematics and physics were still very permeable. His great contributions to the development of mathematical physics, e.g. to special relativity theory, are discussed elsewhere, e.g., see [88], and the present paper focuses on dynamical systems. Evidently there are many interconnections between these various fields. Poincaré has been extremely generous with his ideas and his legacy is large and fruitful, a legacy from which the current discipline of dynamical systems greatly profits. The beginning of these developments consisted of the theory of ordinary differential equations, mainly applied to problems related to celestial mechanics.

One main, general characteristic of Poincaré's approach to any of his fields of interest, is his open-mindedness to all kinds of mathematics and his readiness to connect these several fields. Regarding dynamical systems, he extended the then dominant calculus- and analysis driven-approach by including geometry and algebra and also certain considerations close to probability. To some extent this holds for topology as well, although this branch of mathematics was still in its infancy. In fact Poincaré himself has stood at its cradle when developing his 'analysis situs', treated elsewhere in this volume. For reasons of readability in this paper we deliberately maintain an anachronistic style; for a more faithful translation of Poincaré's diction into more modern terminology, we refer to, e.g., [88].

In particular we shall discuss elements of Poincaré's [63, 65] and of his monumental [69], also see [73]. Already in [65] the idea of a phase portrait was introduced, where the attention is not focused on the computation or approximation of one single solution, but where his geometric organization of all solution curves in a state space is being considered. Also we mention the so-called Poincaré map with respect to a transverse section is introduced in [69], restricting to a fixed energy-level, useful for similarly organizing the dynamics of a two-degrees-of-freedom system. (Poincaré sections and maps are generally defined also for dissipative systems: the sections then should have co-dimension 1 in the state space.) Here the concept of homo- or heteroclinic orbit occurs, which can give rise to a phenomenon today called tangle. In the analytic approach, tangle implies the divergence of certain perturbation series, and in later days was related to chaos. Another geometric tool is the index of a singularity of vector fields (now called Poincaré–Hopf index) as well as its relationship with the Euler characteristic of the domain or surface at hand. We also briefly touch upon the theorems of Poincaré–Bendixson and Poincaré–Birkhoff. Although many applications of these ideas occur in the conservative dynamics of celestial mechanics, also the general qualitative phenomenon now called Hopf bifurcation (also called phenomenon of Poincaré–Andronov [4]) forms one of the geometric discoveries. The Poincaré recurrence theorem [68] and the relationship with statistical mechanics will also be discussed.

Celestial mechanics

According to Leonardo da Vinci mechanics is the paradise of mathematicians. Between 1700 and 1900 this assertion surely holds for celestial mechanics, which during four centuries has exerted an enormous impact on the development of science and technology.

Remarks.

– The pièce de résistance of mathematical power in celestial mechanics surely was

Poincaré in his study

the discovery of Neptune in 1846, predicted from aberrations in the orbit of Uranus. In these Newtonian computations Adams and Leverrier played a role [46], while also Bessel made some contributions.

- To illustrate the accuracy of astronomical observations and computations we mention the perihelion precession of Mercury, established at a value of $5600'' = 1.5556^0$ per century. Computations have shown that, within the classical Newtonian mechanics, the perturbations of the other planets account for $5557'' = 1.5436^0$ per century. The difference of $42.98''$ can be explained by Einstein's general theory of relativity [53]. This amount of accuracy was already attained around 1900.

Stability of the solar system

Since Newton, Euler and Laplace a central question has been whether the solar system as witnessed and reported by mankind is 'stable' in the sense that the observed multiperiodic motion will persist for ever. Here one may think of the effectiveness of the Gregorian leap year convention (give or take a corrective leap second now and then) or the prediction of the Easter dates and ask how all this works out in the perpetually long run. The solar system itself is quite large and subsystems of it often have been described in terms of three bodies: say the Earth-Moon-Sun or a Sun-Jupiter-asteroid system. Moreover, mathematical simplifications exist like the Restricted Planar Three Body Problem (RPTB), where two primary bodies move uniformly in circles and where the mass of the third body is considered to be small enough not to effect the motion of the primaries and where everything takes place in one fixed plane.

The main mathematical techniques for a long time came from perturbation theory: that is the analysis of a given motion as a perturbation of a more well-known one, say, in terms of one or several Kepler ellipses. One major tool here consists of series expansions.

Analytic methods, normal forms

Poincaré's thesis [63] contributed a lot to this classical perturbation theory. One part of his legacy consists of a local normal form theory, where the series expansions of a vector field (system of ordinary differential equations) in a equilibrium is duly simplified. For later developments also compare Birkhoff [13], developments that extend till today. This simplification facilitates the perturbation analysis based on the series, where the simplified (or normalized) lower-order part forms the unperturbed system and the higher-order terms the perturbation. One important aspect of the simplification often is given by a toroidal symmetry that turns the lower order part of the series into an integrable system. The integrable approximation in turn can be reduced to lower dimensions as is common in classical mechanics, for a more modern point of view compare Arnold [3]. As we shall see later, such series generically do not converge. (Generically they diverge, where the space of real analytic systems is endowed with the compact-open topology on holomorphic extensions [19, 35, 74, 83].) Needless to say however, they always can serve asymptotic purposes. (At the time this was a controversial issue: convergence was considered essential by mathematicians like Cauchy and Weierstraß.)

In [69] a similar problem is addressed, but now the aim is to continue a periodic solution in terms of a perturbation parameter, for Poincaré's comments on these Lindstedt series see [66–67]. Nowadays one generally speaks of Poincaré–Lindstedt series. Convergence of such a series in general can be obtained as an application of the implicit function theorem or similar methods.

Remarks.

- I like to mention the celestial mechanics computations by De Sitter on the $1 : 2 : 4$ orbital resonance of the Galilean Jupiter satellites Io, Europa and Ganymede, that leads to a libration in their motion. This phenomenon was observed by David Gill (Cape Town) and De Sitters computations started around 1900 leading to a PhD thesis under Kapteyn. Later he again spent a lot of time on this subject [81–82], using Poincaré's normalized resonant series expansion [63, 69]. (It seems that De Sitter himself esteemed this celestial mechanics project higher than his work on a global solution for the equations of general relativity, later on coined as the De Sitter universe.)
- For more historical details and references see the very readable De Sitter biography by Guichelaar [29].

Example 1 *(Holomorphic linearization).* To fix our thoughts mathematically, we consider the problem of holomorphic linearization that runs as follows, see [64]. Also see [4, 52]. Given is a holomorphic germ $F : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$F(z) = \lambda z + f(z), \tag{1}$$

where $f(0) = f'(0) = 0$ and the question is whether a bi-holomorphic diffeomorphism $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ exists, such that

$$\Phi \circ F = \lambda \cdot \Phi, \tag{2}$$

i.e., that linearizes F. Here λ is a complex parameter. A brief computation shows that a formal solution $\Phi(z) = z + \sum_{j\geq 2} \phi_j z^j$ exists in terms of the Taylor series of f, if and only if $\lambda \neq 0$ and $\lambda \neq e^{2\pi i p/q}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, i.e., if λ is not a root of unity. Indeed, it easily can be shown that the coefficient ϕ_n

contains a factor

$$\frac{1}{\lambda(1-\lambda)\cdots(1-\lambda^n)}. \tag{3}$$

The question then is whether or not the series converges. In the hyperbolic case $0 < |\lambda| \neq 1$ this question is answered affirmatively by Poincaré [64]. For an elaborate discussion, including Poincaré's elegant, geometric solution of this hyperbolic case, see [4]. In the elliptic case where λ belongs to the complex unit circle, the exclusion of λ from being a root of unity is not enough for convergence. Indeed, even in this case the denominators in (3) accumulate on zero, which forms an example of the notorious small divisor problem. Sufficient for convergence are Diophantine conditions on λ, which require that for positive constants τ and γ inequalities

$$\left|\lambda - \mathrm{e}^{2\pi\mathrm{i}\frac{p}{q}}\right| \geq \frac{\gamma}{q^{\tau}}, \tag{4}$$

hold true for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The corresponding subset of λ's in the complex unit circle is both meagre and of full measure. This solution was given by Siegel in 1942, compare with [80]. In his 1969 thesis Bruno extended this sufficient condition in terms of the decay rate of continued fractions; the latter Bruno condition was proven also necessary by Yoccoz [91], for which he received a 1994 Fields medal. This is a highly successful part of mathematical research, that surely belongs to Poincaré's mathematical legacy.

Small divisors

In the conservative dynamics of classical mechanics, including celestial mechanics, small divisor problems occur a lot. As in the above example this phenomenon is related to a dense set of resonances where a certain series is not even defined. The situation can be described as follows, for instance by Moser [57]. In the n-dimensional space of frequencies $\omega_1, \omega_2, \ldots, \omega_n$ a resonance occurs if for some integer vector $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \setminus \{0\}$ the relation

$$k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n = 0 \tag{5}$$

holds. For fixed $\mathbf{k}$ equation (5) defines a hyperplane in the frequency space and the union of these planes over all $\mathbf{k} \in \mathbb{Z}^n \setminus \{0\}$ forms the dense resonance web. Certain perturbation series of the form

$$\sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \frac{c_{\mathbf{k}}}{k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n} \cdot \mathrm{e}^{2\pi\mathrm{i}(k_1x_1+k_2x_2+\cdots+k_nx_n)},$$

show up, where the constants $c_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^n$ a priori only satisfy a certain decay condition as $|\mathbf{k}| \to \infty$. As in Example 1 exclusion of the resonances gives formal existence of the series, but convergence still is problematic due to small divisors.

Poincaré recognised the problem that was solved later by Kolmogorov [41–42] and many others, see below. This problem concerns the persistence of quasi-periodic motion as these occur in simplified approximations in which, e.g., uncoupled Keplerian motions show up. This perturbation problem was at the heart of the contest called by king Oscar II of Sweden.

Remarks.

- This history is world-famous, for a very readable and historically detailed account see June Barrow-Green [10]. Summarizing we recall that Poincaré's essay, although containing a wealth of ideas, contained one essential mistake, namely the claim that the perturbation series do converge.
- Poincaré discovered (triggered by a query of Phragmén) and repaired the mistake, concluding that the series have to diverge for a significant set of frequencies. Mittag-Leffler let him use the prize money to buy the volumes of the *Acta Mathematica* that were already published. His divergence proof, published in the article [68] in the same journal, uses geometric methods that we will address now.

Tangle and divergence

The restricted planar three body (RPTB) problem ends up with a two-degrees-of-freedom Hamiltonian system, which lives in a four-dimensional state space. Then, restricting to a three-dimensional energy hypersurface we can consider a section (or slice) transverse to the dynamics. Following the integral curves of the system in turn gives rise to a two-dimensional Poincaré map, compare the sketch in Figure 1 for some orbits. In the present setting of classical mechanics such a map preserves area. If the perturbation series at hand were convergent the orbits of this map would nicely trace the integral curves of a vector field. The presence of tangle however prevents this convergence, as Poincaré himself concluded, compare with, e.g., Takens [84].

What then is tangle? To understand this we recall that a hyperbolic fixed point p of a map has both a stable manifold $W^s(p)$ and an unstable manifold $W^u(p)$, which in the present planar case are smoothly immersed curves. The points of $W^s(p)$ converge to p under forward iteration and similarly the points of $W^u(p)$ by backward iteration. For two of such saddle points p_1 and p_2 it may happen that the intersection $W^u(p_1) \cap W^s(p_2)$ is nonempty. The intersection consists of point heteroclinic to p_1 and p_2. In the case where $p_1 = p_2$ the intersection points are called homoclinic (Poincaré himself speaks of double-asymptotique [88]). For a vector field, by uniqueness of solutions, this would imply coincidence of $W^u(p_1)$ with $W^s(p_2)$. For a map however, this intersection generically is transverse, again see [19, 35, 74, 83]. In Figure 1 a sketchy impression is given of tangle, for a realistic impression how the tangle explodes near a saddle point see Figure 3 (right).

Figure 1 Hetero- and homoclinic 'tangle' in the stable and unstable manifolds of a periodic orbit, sketched for a planar iso-energetic Poincaré map as this occurs in a suitable three-body problem.

From that time on the mainly analytical investigation of dynamical systems has been extended by considering the geometrical organisation of the entire state space (or phase space). This involved the study of equilibria, periodic solutions, invariant manifolds like stable and unstable manifolds and invariant tori. At the same time the mathematical assertions became more qualitative.

Geometric ideas

We introduce a few geometric ideas and results introduced and obtained by Poincaré. This includes the rotation number and the Poincaré–Hopf index theory for equilibrium points of vector fields in the two-dimensional case; the higher dimensional case was added later by Heinz Hopf. Also we deal with the Poincaré–Birkhoff and the Poincaré–Bendixson theory.

Rotation number of circle homeomorphisms

The rotation number for an orientation preserving homeomorphism of the circle $\Phi : \mathbb{S}^1 \to \mathbb{S}^1$, first considered in Poincaré [65], is defined as the average rotation

$$\varrho(\Phi) = \frac{1}{2\pi n} \lim_{n\to\infty} \left(\tilde{\Phi}^n(x) - x\right) \mod \mathbb{Z},$$

where $\tilde{\Phi} : \mathbb{R} \to \mathbb{R}$ is a lift of Φ. This notion depends neither on the point $x \in \mathbb{S}^1$ nor on the choice of the lift $\tilde{\Phi}$ and moreover is invariant under topological conjugation.

Rationality or irrationality of the rotation number. If Φ has a periodic orbit of (prime) period q, then $\varrho(\Phi) = p/q$ for some p coprime with q. Conversely $\varrho(\Phi)$ is rational implies that Φ has a periodic orbit. On the other hand, if $\varrho(\Phi)$ is irrational and Φ is of class C^2, then Φ is topologically conjugated to a rigid rotation. These results go back to Denjoy [21]. For details also see Devaney [22] and Nitecki [59].

In the case where the rotation number is Diophantine there exist smooth conjugations with rigid rotations, compare with Example 1. The perturbative version of this by now is a standard result in Kolmogorov–Arnold–Moser theory as described further on, but the non-perturbative case certainly is not, see Herman [33–34].

The Poincaré–Birkhoff theorem on periodic orbits. The Poincaré–Birkhoff fixed point theorem fits in the theory of area preserving planar maps as met before, compare with the settings of Figures 1 and 3. Consider such a planar map Φ with two invariant circles with rotation numbers $\varrho_1 < \varrho_2$, bounding an annulus. The Poincaré–Birkhoff theorem then asserts that for any rational number p/q with $\varrho_1 < p/q < \varrho_2$ there exists a Φ-periodic orbit of rotation number p/q. Also see the contribution of Verhulst to this volume. For proofs see Poincaré [72] and Birkhoff [12]. A simple proof can be given in the case where Φ is a perturbed twist-map, see Birkhoff [13]. See Arnold et al. [5] and Moser [56] for presentations of this proof based on the implicit function theorem. For further developments we refer to [31–32, 40].

Poincaré–Hopf index theory

This topic connects the possible dynamics of any system to the global topology of the manifold on which it is defined, see [65]. Consider a vector field X on a closed surface M. If $p \in M$ is an isolated equilibrium point (also called rest point or singularity) of X we can define the index $\text{ind}_p X$ by taking any small simple curve $t \in \mathbb{S}^1 \mapsto \gamma(t) \in M$ around p and considering the well-defined direction field $1/(|X(\gamma(t))|) \cdot X(\gamma(t))$. Here $\mathbb{S}^1$ denotes the unit circle. This induces a map $\mathbb{S}^1 \to \mathbb{S}^1$ and we define $\text{ind}_p X$ as its winding number.

Remarks.

- Locally one can define the closed 1-form

$$đ\varphi = \frac{X_2 \, \mathrm{d} X_1 - X_1 \, \mathrm{d} X_2}{X_1^2 + X_2^2},$$

where

$$X(x_1, x_2) = X_1(x_1, x_2)\partial_{x_1} + X_2(x_1, x_2)\partial_{x_2} \, ;$$

in this case we can express $\text{ind}_p X = \frac{1}{2\pi} \oint_\gamma đ\varphi$, compare with [2, 51].
- This definition was generalized to higher dimension by Heinz Hopf, using Brouwer's notion of degree, see [38].

There is an interesting connection with the global topology of the surface, given by the formula

$$\sum_p \text{ind}_p X = \chi(M),$$

where $\chi(M)$ is the Euler-characteristic of M, also compare with Lakatos [44]. One consequence of this is the well-known 'hairy ball' theorem that any vector field X on the 2-sphere $M = \mathbb{S}^2$ must have at least one singularity since $\chi(\mathbb{S}^2) = 2$.

The Poincaré–Bendixson theorem

Another interesting result [65] concerns the surfaces $M = \mathbb{R}^2$ or $M = \mathbb{S}^2$, dealing with the possibilities of the asymptotic dynamics as time goes to infinity. The corresponding set is the ω-limit set of a point $p \in M$ given by

$$\omega(p) = \{y \in M \mid \lim_{j\to\infty} \Phi^{t_j}(p) = y \text{ for a sequence } \{t_j\}_{j=1}^\infty \text{ with } \lim_{j\to\infty} t_j = \infty\}.$$

The Poincaré–Bendixson theorem asserts that on $M = \mathbb{R}^2$ or $M = \mathbb{S}^2$ the ω-limit sets of vector fields with a finite number of equilibria can only be

- equilibria,
- limit cycles,
- or graphs of equilibrium points and their stable and unstable manifolds,

see Poincaré [65] and Bendixson [11]. This result often is used to prove the existence of a (i.e., at least one) limit cycle in the absence of any equilibria. Compare with Palis and De Melo [61] or with Verhulst [87].

Multi- or quasi-periodicity

Multi-periodic and chaotic dynamics have obtained a lot of interest in the second half of the 20th century, an interest that surely belongs to the legacy of Poincaré. A central role was played by Kolmogorov [41–42]. To explain this we again address the stability of the solar system, the prize essay question.

Integrable systems

Multi- or quasi-periodic dynamics were very well-known from the completely integrable systems of classical mechanics. In a Hamiltonian system with n degrees of freedom an open and dense part of the entire state space then is foliated by n-dimensional Lagrangian invariant tori. Examples are given by systems of uncoupled Keplerian motion. Locally an integrable system is described by action-angle coordinates $(I, \varphi) = (I_1, I_2, \ldots, I_n, \varphi_1, \varphi_2, \ldots, \varphi_n)$, in which the symplectic form reads $\mathrm{d}I \wedge \mathrm{d}\varphi = \sum_{j=1}^n \mathrm{d}I_j \wedge \mathrm{d}\varphi_j$. Compare with Arnold [3]. The Lagrangian tori are parametrized by I varying over an open piece of $\mathbb{R}^n$. (The entire union of Lagrangian tori forms a bundle, for a nice description see Duistermaat [23].)

The motion on such a torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ then is generated by a constant vector field

$$\begin{aligned} \dot\varphi_1 &= \omega_1 \\ \dot\varphi_2 &= \omega_2 \\ &\vdots \\ \dot\varphi_n &= \omega_n \,, \end{aligned}$$

or $\dot\varphi = \omega$ for short, where $\omega = \omega(I)$. Usually such a motion is called conditionally periodic or multi-periodic [3, 18]. Here the resonance web (5) again plays a role. Indeed, in the complement of the web any individual integral curve densely fills the invariant n-torus that it is part of. Usually this form of multi-periodic is called quasi-periodic. Within the web the n-tori are foliated by lower-dimensional invariant tori with similar dense orbits and at n-fold resonance points the torus is foliated by periodic orbits.

The matter of stability of the solar system has boiled down to the question to what extent this geometric structure persists under small perturbations that destroy the toroidal symmetry. A simple-minded, yet significant, setting of this problem runs as follows. A

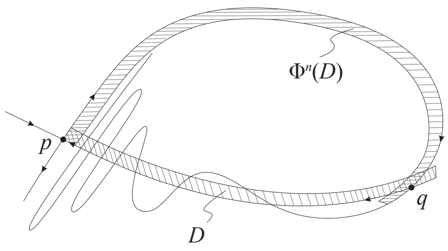

Figure 2 Two-dimensional diffeomorphism Φ with a saddle point p and a transversal homoclinic intersection q. The set D, homeomorphic to the unit square, by a well-chosen iterate Φ^n covers D in a horseshoe-like way.

completely integrable approximate system is formed by considering the Newtonian planetary motion around the sun, neglecting the interactions between the planets. This approximation is formed by the uncoupled dynamics on a number of Keplerian ellipses. One first obstacle to prove stability for the full system, i.e. including the interaction between the planets, is formed by rather strong orbital resonances that occur in the integrable approximation, which has led to a similar but more sophisticated set-up.

Kolmogorov–Arnold–Moser theory

The persistence problem of tori in nearly integrable Hamiltonian systems again has to deal with small divisors, it is basically the same problem that Poincaré faced in his [68–69], also compare with [80]. We present the solution given by Kolmogorov [41–42].

Given suitable constants $\tau > n - 1$ and $\tau > 0$, Diophantine non-resonance conditions are introduced of the form

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{\gamma}{|\mathbf{k}|^{\tau}}, \tag{6}$$

for all $\mathbf{k} \in \mathbb{Z} \setminus \{\mathbf{0}\}$, compare with (4). This defines a closed, nowhere dense set of positive Lebesgue measure, tending to full measure as $\gamma \downarrow 0$. Moreover a non-degeneracy condition — later named after Kolmogorov — is introduced, requiring that the frequence map $I \mapsto \omega(I)$ is a local diffeomorphism.

Kolmogorov's theorem now roughly reads as follows. Given Kolmogorov non-degeneracy, for sufficiently small perturbations of the integrable system, the Diophantine tori are persistent, their union forming a subset of the state space having positive Liouville measure.

Remarks.

- Refinements and further proofs have been added by Arnold [1] and Moser [54], also see [5, 7] and [55–58], for which reason we now speak of Kolmogorov–Arnold–Moser (or KAM) theory. A similar result holds when restricting to a fixed level of energy, this is the iso-energetic KAM theorem.
- In fact the quasi-periodic motion on the unperturbed and perturbed tori is smoothly conjugated, where in the I-direction smoothness has to be interpreted in the sense of Whitney [89].
- The KAM theory extends to other classes of systems, such as the general dissipative class, reversible or volume-preserving systems, etc., see [55], for an overview and many references see [16–17].

A second obstacle for proving the stability of the solar system in this way, is formed by the fact that the perturbations are far too large for applications of KAM theory. Recently rumours go that the inner solar system might very well be chaotic, see Laskar et al. [43]. Based on an estimate of Lyapunov exponents the first major disasters in this respect may be expected on a time-scale of the order of 100 million years... This is relatively soon as compared to the age of the solar system.

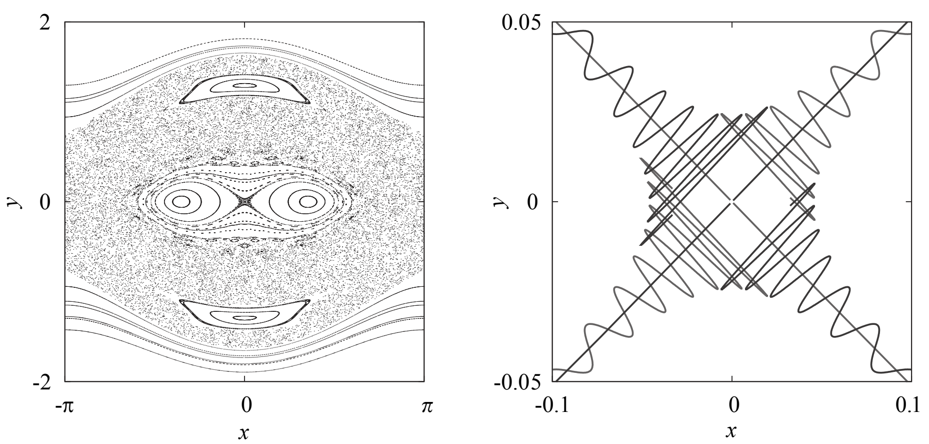

Figure 3 Chaos in a model of the swing in a 1 : 2 resonance, e.g., see [18]. Left: Iterates of a Poincaré map (or stroboscopic map) are shown. The invariant circles are within reach by KAM theory. The large cloud of points is part of one single 'chaotic' orbit, which is related to tangle and Smale horseshoes. Right: a few 'switches' of the tangling separatrices near the central, unstable fixed point.

Tangle and the Smale horseshoe

Homoclinic tangle leads to chaos. This is an anachronism, but the description that Poincaré himself gives of Figure 1 explains that he must have had an insight in the enormous complexity of a generic area preserving map ([69], Vol. III):

"Que l'on cherche à se représenter la figure formée par ces deux courbes [...] On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problême des trois corps et, en général, de tous les problèmes de dynamique où il n'y a pas d'intégrale uniforme et où les séries de Bohlin sont divergentes."

In fact it is this complexity that prevents the series from converging, compare with Takens [84]. Throughout the 20th century this phenomenon recurred in various examples, where several forms of symbolic dynamics were introduced to describe this, starting with Birkhoff [13] and Cartwright–Littlewood [20]. In the 1960s the topologist Smale [83] introduced a universal model in the form of the horseshoe map, see Figure 2. Often it takes a sufficiently high iterate of the map to turn heteroclinic points, e.g. as in Figure 1, into homoclinic ones. This largely reduced the problem to geometry and symbolic dynamics. The presence of horseshoes already implies a weak form of chaos, since the topological entropy is non-zero [5, 49]. At the end of this paper we will say more about chaos.

Invariant measures

The phase flow of an n degree-of-freedom Hamiltonian system conserves the Liouville volume, which is the nth wedge-power of the symplectic form, compare with [3]. Similarly the restriction of this flow to an energy hypersurface conserves the conditional $(2n - 1)$-dimensional volume. The corresponding iso-energetic Poincaré maps from Figures 1 and 3 inherit a natural area form.

One consequence of measure-preservation is that no point attractors can occur. Below we first discuss the Poincaré recurrence theorem and next indicate a few links with statistical mechanics, where also the subject of ergodicity will come to pass. An important part of the current status quo in the area of dynamical systems rests on these theories, compare with Ruelle [75–76]. This part of Poincaré's legacy runs by Kolmogorov and Sinai [78].

The Poincaré recurrence theorem

The Poincaré recurrence theorem stems from [68], for the formulation below see also [3]. Let D be a set of bounded volume and consider a map $\Phi : D \to D$ that preserves volume. Then for any point $x \in D$ and any neighbourhood U of x there exists a point $y \in U$ such that the iterate $\Phi^n(y) \in U$ for some $n > 0$. A proof can be given as follows. In the infinite sequence

$$U, \Phi(U), \Phi^2(U), \ldots,$$

of sets all have the same volume. Since D has bounded volume, there must be some $k \geq 0$ and $\ell \geq 0$ with $k \neq \ell$, say with $k > \ell$, such that

$$\Phi^k(U) \cap \Phi^\ell(U) \neq \emptyset,$$

which implies that $\Phi^{k-\ell}(U) \cap U \neq \emptyset$. From this the assertion follows by taking $n = k - \ell$ and $y \in \Phi^{k-\ell}(U) \cap U$.

Repeating the argument with U replaced by $\Phi^{n_1}(U)$ with $n_1 = k - \ell$ gives an infinite sequence of instants $n_1, n_2, n_3, \ldots$, for which the iterate comes close to the initial point x.

Remarks.

- Periodic and quasi-periodic motions are recurrent as well as are certain forms of more complicated motions, like ergodic or chaotic motions. See below.
- Recurrence times can be extremely long, a great many orders of magnitude longer than the age of the universe, also see [86]. Below we shall return to this.

Links with statistical mechanics

Boltzmann was a pioneer in statistical mechanics and thermodynamics [14], where he introduced probability in many-particle systems that model gases and fluids. In the ensuing thermodynamics macroscopic quantities emerge like energy, temperature, pressure, entropy, where the latter is a measure for the disorder of the dynamics. The first law of thermodynamics is conservation of energy, while the second concerns the increase of entropy. The relation of the second law with dynamics was puzzling. From the beginning on a leading question has been to what extent thermodynamics is compatible with classical mechanics. Two themes play a special role in this development, namely irreversibility (leading to the well-known 'arrow of time' [24]) and ergodicity.

Reversibility. In general classical mechanics is invariant under time-reversal, while thermodynamics is not as a consequence of the second law. This controversy was already noted by Loschmidt [48], the so-called 'Umkehreinwand', for continuation of the discussion see Boltzmann [15]. The answer to the problem lies in statistics: reversal is possible, but highly improbable. (This also 'explains' the extremely long recurrence times noted before. For later considerations see [45].) Zermelo [92] moreover observed that Poincaré recurrence gives an explicit obstruction to irreversibility; this is the so-called 'Wiederkehreinwand'. We now quote Poincaré himself as follows [70], indicating that he was not convinced by these applications of his recurrence theorem:

"Qu'une goutte de vin tombe dans un verre d'eau; quelle que soit la loi du movement interne du liquide, nous verrons bientôt se colorer d'une teinte rosée uniforme et à partir de ce moment on aura beau agiter le vase, le vin et l'eau ne paraîtront plus pouvoir se séparer. (...) Tout cela, Maxwell et Boltzmann l'ont expliqué, mais celui qui l'a vu plus nettement, dans un livre trop peu lu parce qu'il est difficile à lire, c'est Gibbs dans ses principes de la Mécanique Statistique."

Ergodicity. Ergodicity of a particle system also goes back to Boltzmann [14]. Heuristically it expresses that all possible states of the system will be approximated from virtually any given initial state. A mathematical characterisation is that a measure preserving system with evolution $\Phi^t, t \in \mathbb{R}$, is ergodic whenever no disjoint, Φ-invariant subsets A and B of positive measure exist. This indeed expresses that the dynamics of the (particle) system is quite mixing. For stronger mixing conditions also see the ensuing work on ergodic theory of dynamical systems [5, 13, 49, 78].

In statistical mechanics often ergodicity is assumed per hypothesis, restricting to an energy level set. The existence of further integrals, e.g., due to additional assumptions like symmetry, would form an obstruction to the ergodic hypothesis. It was already noted early that stability of the solar system, in the sense that integrals exist, would provide a counterexample. Later it was proven that generically in the dynamics of classical mechanics, apart from the energy, no further integrals occur [19, 74]. However by definition near-integrability is a persistent property.

Kolmogorov [42] (we recall his 1954 closing address of the IMC in the Amsterdam Concertgebouw) noticed a far more serious obstruction to the ergodic hypothesis when initiating KAM theory, see above. Indeed, the persistent occurrence of a union of invariant quasi-periodic of positive measure per energy level [3] leads to a generic violation of ergodic hypothesis in classical mechanics. It seems, however, that this particular obstruction is diminishing when the number of degrees of freedom increases, e.g., see [39].

Edward Norton Lorenz (1917–2008)

Remarks.

- As reported in [30] (a book review is contained in this same volume) there was an interaction between Postma and Poincaré [71] on fluctuations in the entropy where the notions 'entropie grossière' (or 'coarse grained entropy') and 'entropie fine' play a role. The former of these is related to the modern concept of Markov partition [18, 40].
- It should be mentioned here that Einstein's paper [25] on Brownian motion

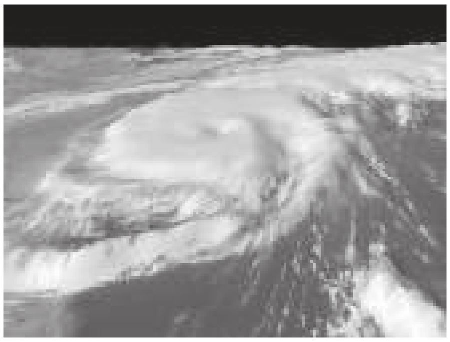

Figure 4 An enormous input to 'chaos theory' came from fields like meteorology; here we see a hurricane.

has a predecessor in Bachelier's PhD thesis [9], written under supervision of Poincaré on an analogous piece of mathematics, applied to financial problems.

– Regarding special relativity there is some controversy on priorities between Lorentz and Poincaré on the one hand, and Einstein on the other. None other than Whittaker [90] attributes special relativity to Lorentz and Poincaré. Also see [88].

Chaos

From the 1970s on the idea of chaos, as a possible property of non-linear dynamical systems that implies a fundamental unpredictability of evolutions, became increasingly important. Needless to say that the development of this part of the theory also belongs to the Poincaré legacy. In a further characterization of chaos an invariant measure comes into play, for instance the measures related to the Liouville volume as discussed before and as these show up in conservative dynamics. In fact one characterisation then is whether the system is ergodic with respect to this. In Figure 3 (left) the large cloud is part of one orbit and one long-standing conjecture is that this orbit densely fills a set of positive area on which the dynamics is ergodic [5].

Remarks.

– Also in the world of dissipative systems chaos has become important. Triggered by Lorenz's ground-breaking paper [47], also see Figure 4, and other examples, the idea of strange attractor [77] came up on which later on invariant measures were constructed which led to ergodicity as a characterization of chaos. One key notion in this respect is the Sinai–Bowen–Ruelle measure, for details see [17–18, 31–32, 40, 78–79].

– Related to this development dynamical invariants were developed, like entropy (both topological and metric i.e. measure theoretical), Lyapunov exponents, etc., that also are important for application in concrete models, also see [26, 60, 74, 87–88].

Concluding remarks

Inevitably in a rather short paper like the present one it is not possible to give a complete description of Poincaré's legacy in the theory of dynamical systems. We have focused on the lines via Birkhoff, Kolmogorov, Sinai and Arnold, for instance tacitly passing by the Russian schools of Lyapunov, Bogolyubov and Andronov. For supplementary information the reader is referred to [27]. After Siegel and Moser the emphasis on celestial mechanics was also diverted to other fields. Smale and Thom [83, 85], both Fields medal winners on variations of the Poincaré conjecture, introduced the concepts of transversality and genericity in the field of dynamical systems. (The Poincaré conjecture is treated elsewhere in this volume. The conjecture was recently solved by the Russian mathematician Perelman.) Among many other things they had a profound influence on the systematic study of singularity theory and bifurcations [6, 8, 85], for another link with chaos also see Palis and Takens [62].

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The interaction of geometry and analysis in Henri Poincaré's conceptions

Henri Poincaré had many interests, both inside and outside science. His special attention in this was devoted to the interaction between different fields of knowledge. In this article Ferdinand Verhulst goes into the interaction between mathematical disciplines, where he concentrates on geometry and analysis.

To appreciate the enormous amount of scientific results obtained by Henri Poincaré (1854–1912), it helps to realise how diverse and important the various influences were on the activities of his mind. His talents were both in the humanities and in science. When he was young he wrote hundreds of letters to his family and friends, mixing them with his own poetry; he got honourable mentions for his essays at school, and he also wrote a novel. In psychology, his observations of how the human mind works to create and invent became textbook accounts. Later in life he became known to the general public because of his philosophical essays. In science, his interests were in geology, physics and astronomy, and of course mathematics. All these fields influenced his ideas and work. It is not surprising that also in mathematics his creative attention was directed at a great many different topics. In this article we will be concerned with two mathematical disciplines, geometry and analysis, the historical development of their interaction and their role in Poincaré's mathematics.

Geometry and the rise of modern science

The rise of modern science in the seventeenth century cannot be dissociated from mathematical thinking and the mathematisation of the description of natural phenomena, called mathematical modeling. The mathematical thinking and writing in that period had a strong geometrical flavour. The *Principia* [11] of Isaac Newton (1643–1727) is not only full of geometric arguments and pictures, but contains actually many geometric theorems. A similar characterization applies to the fundamental treatise by Christiaan Huygens (1629–1695) *Traité de la lumière* [6]. Many examples of geometric reasoning can be found in the work of other prominent scientists, for instance Johann Bernoulli (1667–1748).

Although the dominance of traditional geometric formulations posed a certain restriction, modern science started to develop. Poincaré notes that with Newton, a change of language took place, which enabled 'natural philosophers' to expand science dramatically. Generalization became then possible which had not been guessed before:

"When Kepler's laws were replaced by Newton's, one knew only elliptic motion. Regarding this motion, the two laws differ in form only, going from one to the other is simply by differentiation. From Newton's laws, however, one could by immediate generalization deduce all perturbation effects in celestial mechanics as a whole. If, on the other hand, one would have kept the formulation of Kepler, one would not have considered the orbits of perturbed planets, these complicated curves of which nobody has written down the equations." [17, essay 'L'analyse et la physique']

In Kepler's laws one saw the geometrical and dynamical movements of the planets in elliptical orbits. After the formulation of

Newton's laws one started to use differential equations which enabled us to describe much more complicated gravitational systems than the two-body problem. It is the change of languages that opens up new generalizations which had not been seen before.

Two mathematical cultures

The beginning of the nineteenth century saw the continuation of two cultures, one based on the impressive success of analytic computations and one in which geometric thinking reached a new level. As a striking representative of the analytic culture, Lagrange was very influential. We will discuss Riemann as a representative of geometric thinking.

Lagrange

The use of the analytic ideas of Newton was not always straightforward in practical problems. A simple example is the differentiation of the product of two functions, say $f(x)g(x)$. One can do this by using the definition of differential quotient, but using Leibniz's rule of differentiation $d/dx(fg) = f'g + fg'$ leads quickly to a correct result. The enormous amount of calculations in celestial mechanics and other parts of mechanics in the eighteenth century required such algorithmic rules. To calculate the ephemerides of comets and planets it was essential to have algorithms and more or less 'automatic computation' at one's disposal. A rather extreme example is found in the work of Joseph-Louis Lagrange (1736–1813). Lagrange writes for instance in the introduction to his *Mécanique Analytique* [8]:

"One will not find figures in this work. The methods that I explain therein require neither geometric nor mechanical constructions or arguments, but only algebraic operations forced by regular and uniform steps."

This is clearly seen by Lagrange as an advantage, a way to reach results quickly, and not as a restriction. Are the two volumes of the *Mécanique Analytique*, where all geometry has been weeded out, still of importance? Remarkably enough, the answer is 'yes'. Apart from treating specific problems in mechanics, the general equations of dynamics are derived, the theory of multipliers for extreme values is formulated, the basic ideas of averaging normalization are described in a clear way.

There is no explicit geometry that would have added a synthetic element to the calculations, but the wealth of ideas makes up for this.

The school of Riemann

The successes of analysis in dynamics, in particular in celestial mechanics, had its counterpart in applied mathematics in Germany, but meanwhile geometric thinking went there its autonomous course. This becomes clear in the mathematics of Bernhard Riemann (1826–1866). Poincaré notes in *La valeur de la science* [17]:

"Among the German mathematicians of this century, two names are particularly famous; these are the two scientists who have founded the general theory of functions, Weierstrass and Riemann. Weierstrass reduces everything to the consideration of series and their analytical transformations. To express it better, he reduces analysis to a kind of continuation of arithmetic; one can go through all his books without finding a picture. In contrast with this, Riemann calls immediately for the support of geometry, and each of his concepts presents an image that nobody can forget once he has understood its meaning." [17, essay 'L'intuition et la logique en mathématiques']

These delimitations are not meant to suggest one particular way of mathematical thinking. A few lines later, Poincaré writes:

"The two types of spirits are equally necessary for the progress of science. Both the logicians [doing analysis by successive steps] and the intuitives [doing synthetic geometrics] have done great things which the others could not have achieved. Who would dare to say that he would have preferred that Weierstrass had never written a word, or that there had not been a Riemann? So, analysis and synthesis both have their legitimate part to play."

It is interesting to consider Riemann's papers in the light of Poincaré's remarks.

At the occasion of his 'Habilitation' in Göttingen (1854), Riemann lectured on the foundations of geometry [20], see also [19]. Riemann starts with experience and notes that the Euclidean foundations are not necessary, but that they have an acceptable certainty. He formulates a research plan for n-dimensional manifolds and spaces. Weyl [19] links these considerations with later results in geometry, for instance by Klein, and with general relativity.

The collected works [19] starts with a treatise on the foundations of complex function theory, without figures but, as noted by Poincaré, "each of its concepts presenting an image". The interpretation of a complex function in the neighbourhood of a singularity plays a prominent part.

A long article on Abelian functions in [19] is written in the same style, it contains four figures. This long article discusses the problems of many-valued functions in the context of results from the analysis situs of Leibniz.

Bernhard Riemann (1826–1866)

Comparing the approach of Riemann and Lagrange, it is interesting to consider Lagrange's theory of analytic functions [9]. The book consists of three parts with the first part discussing series expansions for implicitly defined real functions. The second part has a strong geometric flavour; it applies the preceding theory to obtain tangents of curves, curvature calculations and contact problems. There is also an extension to the theory of extreme values and variational calculus. In the third part, finally, the theory of series is applied to problems of mechanics.

In the book of Lagrange on analytic functions we find many calculations to solve a number of particular problems which in the second part are associated with classical geometry. In Riemann's articles, analysis and geometry go hand in hand producing new insights in both fields; an important example is his approach to the many-valuedness of analytic functions near a singularity by introducing a structure of surfaces around it.

Felix Klein

A prominent member of the school of Riemann was Felix Klein (1849–1925). For illustration, we discuss his lectures on ordinary differential equations that started on April 24, 1894 and ended on August 7, 1894, taking place during the so-called 'Sommersemester' in Göttingen. They have been worked out by E. Ritter [7] in 524 handwritten reproduced pages with nice illustrations.

In the preface Klein notes that the present lectures are a natural sequel to his earlier lectures on hypergeometric functions. He also

Felix Klein (1849–1925)

Henri Poincaré, student at the École Polytechnique

mentions that in contrast to other authors he will discuss the global behaviour of solutions, but that this field is so rich that he has to restrict himself to second order linear ordinary differential equations with three singularities. The emphasis in the discussion is on algebraic and transcendental properties of differential equations, oscillation theorems and automorphic functions. The treatment is interesting but was already unusual at that time regarding ordinary differential equations as it is not so much concerned with explicit solutions, but focuses on problems of complex function theory like the role of singularities, Riemann surfaces and uniformisation questions.

After a detailed discussion of algorithmic and synthetic research, Klein states:

“Nowadays one uses everywhere in mathematics again the synthetic method along with the algorithmic method and one can distinguish the problems in the separate disciplines by their treatment according to one or the other. I believe one can weigh the value of both methods against each other: with the algorithmic method, if it can be applied at all, one obtains certainly something, even general comprehensive theorems. This is then not so much the merit of the individual mathematician as he works with the capital of his predecessors, with the supply of ideas which earlier mathematicians have assembled by the creation of the algorithm. It is different with the synthetic method; there everything comes down to having the correct, new thought. There, one does not know whether one finds something, there one has to create one's own path. What one achieves is maybe little, but to a large extent the property of the researcher. The algorithm gives progress in objective respect but not subjectively. One is not so much forced to think independently. The algorithm looks like travelling by train that goes fast and far, but through cultivated landscapes only, the synthetic method is of the settler who with his axe and much trouble penetrates into the jungle and conquers new domains of culture. In any case the second activity must precede the first.

”Klein concludes that in his lectures he will use both algorithmic and synthetic approaches. Algorithmic he considers the treatment of algebraic integrability, including algebraic and transcendental groups, and the theory of Lamé polynomials. Synthetic is the discussion of the oscillation theorem (Sturm–Liouville theory) and the theory of automorphic functions, this last chapter taking nearly hundred pages. It is concerned with the properties of analytic continuation, the relation with the geometry of Riemann surfaces and uniformisation questions.

The mathematical education of Poincaré

In the nineteenth century, geometry in France consisted of the classical Euclidean geometry, supplemented by analytic and projective geometry, including its applications such as descriptive geometry. Henri Poincaré, eighteen years old, enrolled in the special course that would prepare him for the entrance examinations of the ‘Grandes Écoles’, the École Normale Supérieure and the École Polytechnique. Before he began these studies, he plunged into the mathematics textbooks of the time. According to [1] these included *La Géométrie* by Eugène Rouché, *L'Algèbre* by Joseph Bertrand, *Cours d'Analyse de l'École Polytechnique* by Jean-Marie Duhamel and *La Géométrie Supérieure* by Michel Chasles. The last two books on the list are particularly remarkable. Jean Duhamel's calculus book was a textbook for the École Polytechnique, where Duhamel (1797–1872) himself lectured. The geometry book of Michel Chasles is on advanced geometry while proposing a pronounced philosophy.

Michel Chasles advocates a balance

Michel Chasles's geometry text emphasized the complementary roles of analysis and geometry; it was original, difficult, questioned by a number of colleagues, yet written in an engaging style. Chasles (1793–1880) formulated his approach in his 1846 inaugural lecture for the geometry chair at the Sorbonne as follows (see also [4]):

Michel Chasles (1793–1880)

“If one knows that the subject of geometry is the measure and characteristics of space, then one knows how extended this field is, and that one does not even know where the boundaries of this domain end. For space that one imagines changes shape infinitely often, and the features of each of the forms arising in nature or of those the human spirit can imagine, are themselves extremely numerous, one can even say inexhaustible.”

Chasles continued by proposing a drastic change in the way geometry was understood and practised at the beginning of the nineteenth century, for instance under the influence of Lagrange. It emphasized formulas over pictures, analysis over synthesis. Chasles, in contrast, has this to say:

“One can see the respective advantages of Analysis and Geometry: the former leads with the miraculous mechanism of its transformations quickly from the starting point to the point to be reached, but often without revealing the road that was travelled or the significance of the numerous formulas that have been used. Geometry on the other hand derives its inspiration from thoughtful consideration of things and from the ordered arrangement of ideas. It is obliged to discover in a natural way the statements that Analysis could neglect and ignore.”

When Chasles's *La Géométrie Supérieure* appeared in 1852, his vision of the importance of geometry, using of course analysis as a support, was rather revolutionary in France.

Jean-Gaston Darboux (1842-1917)

It seems more than likely that it became an important influence on Henri's way of thinking, especially considering the fruitful combination of analysis and geometry that is typical of his methods, for instance in the quantitative and qualitative theory of dynamical systems that he would later develop. Jean-Gaston Darboux (1842–1917) wrote his doctoral thesis under the supervision of Michel Chasles, Darboux was one of the supervisors of the doctoral thesis of Henri Poincaré. The view of Chasles is mirrored in [17, essay 'L'analyse et la physique'], where Poincaré writes about the analytic and geometric images evoked by the Laplace equation:

"Thanks to these images, one can see at a glance what pure deduction will show only after successive steps."

Examples of fruitful interaction

We will discuss now a few examples of the interaction of geometry and analysis in the mathematics of Poincaré. More details and other examples can be found in [5] and [21].

Automorphic functions

The integration of differential equations leads more often than not to solutions that are defined implicitly. We are then faced with an inversion problem to find the explicit solution. Consider for instance a simple implicit relation in complex variables: $w = z^2$ with inversion $z = \sqrt{w}$; this leads to the well-known problem that, starting, say on the real axis, and moving on a circle around the origin (the singularity), will produce a different value when arriving again at the real axis. An ingenious solution for the problem of many-valuedness to obtain unique continuation of such a function was proposed by Riemann. Using several surfaces when moving around the singularity and joining them, one obtains the so-called Riemann surface. In the example of the quadratic equation above, one needs two surfaces to be joined. For more general algebraic implicit equations, one needs for such an inversion a finite number of surfaces and so a more complicated Riemann surface.

The German mathematician Fuchs (1833–1902) considered a second order, linear, ordinary differential equation of the form

$$y'' + A(z)y' + B(z)y = 0$$

with A and B holomorphic functions of the complex variable z in a region S. There are two independent solutions $y_1(z)$ and $y_2(z)$ and Fuchs started to consider the ratio $\eta = y_1/y_2$. He was interested in the behaviour of the solutions near singular points of A and B and performed analytic continuation of $y_1(z)$ and $y_2(z)$ along a closed curve around such a singularity and inversion of the function $\eta(z)$. This led him to consider a certain linear transformation of η and, more in general, to look for functions that are invariant under a substitution of the form

$$z \to \frac{az+b}{cz+d},$$

with complex coefficients. So we have

$$f\left(\frac{az+b}{cz+d}\right) = f(z).$$

The ratio of the solutions η should be invariant under these linear substitutions which is a more general property than periodicity. This idea of Fuchs inspired Poincaré to put the idea at a higher level of abstraction. He called the functions with this property Fuchsian, they are now called automorphic. For his subsequent analysis, Poincaré had to distinguish between continuous and discontinuous transformation groups. He understood by a flash of intuition that the continuation of these complex functions, the use of Riemann surfaces and transformations in the complex plane correspond with geometric structures that can be understood only in terms of non-Euclidean geometry. In fact, until Poincaré looked at these problems, non-Euclidean geometry was considered as an artificial playground without much relevance to mathematics in general.

The theory led Poincaré to the formulation of uniformisation problems. The integration of algebraic functions, also obtained from differential equations, and their analytic continuations produce multi-valued analytic functions. Uniformisation of such functions corresponds with obtaining a parametrisation by single-valued meromorphic functions. The development has led to the relation between complex function theory and hyperbolic geometry, and also to many results in the study of quadratic forms and arithmetic surfaces. The theory of uniformisation contains still many fundamental open questions.

Balayage for gravitational potential problems

When considering the Newtonian attraction properties of bodies with a given distribution of mass, one is led to a study of the Laplace and Poisson equations (see *Théorie du potentiel Newtonien* [16]). One of the basic problems is to solve the Laplace equation

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \text{ in } D,$$

with $D \in \mathbb{R}^3$ a bounded domain. If the distribution of mass is given by a function $\rho(x, y, z)$, the corresponding potential is described by the Poisson equation

$$\Delta V = \rho.$$

A twice differentiable function satisfying the Laplace equation $\Delta V = 0$ is called *harmonic*. If we require that on the boundary S of D we have $V = \Phi$ with Φ a known function, this is called the Dirichlet boundary value problem for the Laplace equation. If we would look for solutions on the infinite domain exterior to D with the same boundary condition and certain conditions at infinity, we would have

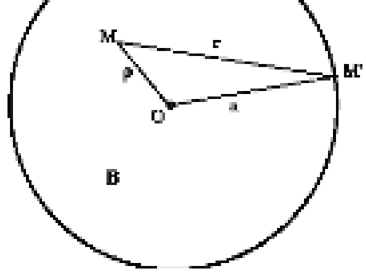

Figure 1 Balayage of a spherical body

the exterior Dirichlet problem for the Laplace equation; this describes the Newtonian gravitational field in the empty space outside a body filling up D; the boundary potential prescribed on S derives from the interior distribution of mass. The exterior Dirichlet problem describes at the same time the electrical force field *outside* a conductor in D with prescribed electrical charge on S.

In the nineteenth century one could solve such problems for simple geometries like spheres and circular cylinders. Existence, uniqueness and actual construction of solutions for more general domains D was a different matter. Today, variational methods like the Dirichlet principle, are useful in this respect, but around 1890, the validity of the variational principles was not yet proved, indeed Weierstrass had thrown doubt on it, so scientists were looking for alternative solution methods. Poincaré invented the so-called balayage or sweeping method, a didactical presentation is given in [16]. In what follows, we describe his theory.

Consider to start with a ball B with spherical surface S, centre O and radius a in three-dimensional space. A point M is located inside the sphere, distance to O is ρ; a surface element $d\omega'$ has centre of gravity M'; the distance from M to M' is r, see Figure 1.

Inside and outside the sphere we have a distribution of mass; at the centre of gravity of a volume element $d\tau$, the density is μ. The gravitational potential V at a point is written as

$$V = V_1 + V_2,$$

with V_1 the potential due to the mass interior to the sphere, V_2 the potential due to the mass exterior to the sphere. As we know from potential theory, the potential V_1 will not change when we replace each mass $\mu d\tau$ in the ball B by a mass layer on the surface S with density in M':

$$\mu' = \frac{a^2 - \rho^2}{4\pi a r^3}\mu d\tau.$$

The mass distribution on the surface S is called the *equivalent layer*. With this procedure we perform a sweeping (balayage) of all the mass in the interior, producing in M' on S the density

$$\mu'' = \int_B \mu \frac{a^2 - \rho^2}{4\pi a r^3} d\tau.$$

The integration is taken over the interior of the sphere. Clearly $\mu'' > 0$. We put

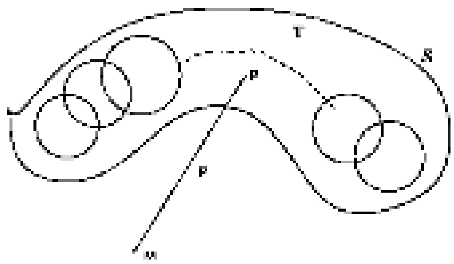

Figure 2 Balayage of a bounded, connected domain T

$$U_1 = \int_S \frac{\mu''}{r} d\omega'.$$

The potential U_1 equals V_1 outside S, $U_1 \leq V_1$ inside S; U_1 is called subharmonic.

Consider now, more in general, a bounded, connected domain T with smooth boundary surface S, see Figure 2. P is a point in T where a mass $m = 1$ is located. A point M in space at distance ρ undergoes a Newtonian attraction force from m determined by the potential

$$V = \frac{m}{\rho}.$$

There exists a Green's function G with respect to the domain T with the properties

$$G = \frac{1}{\rho} - H,$$

where H is harmonic and determined by

$$\Delta H = 0 \text{ in } T \text{ and } H = \frac{1}{\rho} \text{ on } S.$$

Consider now a function V that coincides with H in domain T and equals $1/\rho$ outside the boundary S; V is harmonic in T and outside S, continuous in S and regular at infinity. For a surface element $d\omega'$ as formulated above, the theory of Green's functions produces the potential

$$V = -\frac{1}{4\pi}\int_S \frac{\partial G}{\partial n}\frac{d\omega'}{r}.$$

So we can consider V as the potential of a material layer spread out over S with density

$$-\frac{1}{4\pi}\frac{\partial G}{\partial n}.$$

For a point outside S, the potential from a mass in P equals the potential of the layer on S.

To extend the balayage of a sphere to domain T we use a covering of T by a denumerable set of balls and a corresponding sequence of harmonic functions. In this sequence, each potential U_{n+1} is obtained from the preceding one U_n by balayage. It requires a subtle reasoning for which we refer to [16] and [10].

In modern potential theory one considers domains in $\mathbb{R}^n$ with Borel measures (instead of mass distributions) on sets of a general nature where the balayage produces another suitable measure.

Differential equations and dynamical systems

Poincaré has written extensively about differential equations and dynamical systems, writings that contain a huge amount of material on the interaction of analysis and geometry. We will discuss some examples from two major contributions, the *Mémoire* of 1882 and the *Méthodes Nouvelles de la Mécanique Célèste*.

The Mémoire. Poincaré's first appointment was to mining engineer in Vesoul. This was soon followed by an assignment in mathematics at the university of Caen where he wrote a Mémoire [13] that represented a completely new approach to the study of ordinary differential equations (ODEs). Although the Mémoire is restricted to autonomous 2nd order equations, the research programme sketched by Poincaré for ODEs is very general and at present the programme still dominates research. He writes in the beginning:

"Unfortunately it is evident that in general these equations [ODEs] can not be integrated using known functions, for instance using functions defined by quadrature. So, if we would restrict ourselves to the cases that we could study with definite or indefinite integrals, the extent of our research would be remarkably diminished and the vast majority of questions that present themselves in applications would remain unsolved."

And a few sentences on:

"The complete study of a function [solution of an ODE] consists of two parts:

1. Qualitative part (to call it like this), or geometric study of the curve defined by the function.
2. Quantitative part, or numerical calculation of the values of the function."

Consider the two-dimensional system $dx/dt = X(x, y)$, $dy/dt = Y(x, y)$ with orbits in the (x, y)-phaseplane. For the analysis of the system, Poincaré uses gnomonic projection; this is a cartographic projection of a plane onto a sphere (in cartography of course the other way around). The plane is tangent to the sphere and each point of the plane is projected through the centre of the sphere, producing two points on the spherical surface, one on the Northern hemisphere, one on the Southern. The equatorial plane separates the two hemispheres. Each straight line in the plane projects onto a great circle. So a tangent to an orbit in the plane projects onto a great circle that has one point in common with the projection of the orbit on the sphere. Such a point will be called a *contact*. A point on the great circle in the equatorial plane corresponds with infinity.

The advantage of this projection is that the plane is projected on a compact set which makes it much more tractable. The prize we pay for this is of course that we have to consider with special attention the equatorial great circle which corresponds with the points at infinity of the plane. A bounded set in the plane is projected on two sets, symmetric with respect to the centre of the sphere and located in the two hemispheres.

If in a point (x_0, y_0) we have not simultaneously $X = Y = 0$, (x_0, y_0) is a regular point of the system and we can obtain a power series expansion of the solution near (x_0, y_0).

If in a point (x_0, y_0) we have simultaneously $X = Y = 0$, (x_0, y_0) is a singular point. Under certain nondegeneracy conditions Poincaré finds four types for which he introduces the nowadays well-known names saddle, node, focus and centre. These are called singularities of first type. In the case of certain degeneracies we have singularities of the second type. Points on the equatorial great circle may correspond with singularities at infinity and can be investigated by simple transformations. For instance, if the point is not on the great circle $x = 0$, we transform

$$x = \frac{1}{z},\ y = \frac{u}{z}$$

and consider the transformed equation in z and u. If a point on the great circle $x = 0$ is investigated we transform

$$x = \frac{u}{z},\ y = \frac{1}{z}.$$

The next section of the Mémoire discusses the distribution and the number of singular points. Assuming that X and Y are polynomial and of the same degree and if X_m, Y_m indicate the terms of the highest degree, while we have *not* $xY_m - yX_m = 0$, then the number of singular points is at least 2 (if the curves described by $X = 0$ and $Y = 0$ do not intersect on the two hemispheres after projection, there must be an intersection on the equatorial circle). In addition it is shown that a singular point on the equator has to be a node or a saddle, in the plane one cannot spiral to or from infinity. An important concept to be introduced is *index*. Consider a closed curve, a cycle, located on one of the hemispheres. Taking one tour of the cycle in the positive sense, the expression Y/X jumps h times from $-\infty$ to $+\infty$, it jumps k times from $+\infty$ to $-\infty$. We call i with

$$i = \frac{h-k}{2}$$

the index of the cycle. It is then relatively easy to see that for cycles consisting of regular points one has:

- A cycle with no singular point in its interior has index 0.
- A cycle with exactly one singular point in its interior has index $+1$ if it is a saddle, index -1 if it is a node or a focus.
- If N is the number of nodes within a cycle, F the number of foci, C the number of saddles, the index of the cycle is $C - N - F$.
- If the number of nodes on the equator is $2N'$, the number of saddles $2C'$, the index of the equator is $N' - C' - 1$.
- The total number of singular points on the sphere is $2 + 4n,\ n = 0, 1, \ldots$.

A solution of the ODE may touch a curve or cycle in a point. Such a point is called a *contact*; in a contact the orbit and the curve have a common tangent. An algebraic curve or cycle has only a finite number of contacts with an orbit. Counting the number of contacts and the number of intersections for a given curve contains information about the geometry of the orbits.

A useful tool is the 'théorie des conséquents', what is now called the theory of Poincaré maps. We start with an algebraic curve parametrised by t so that $(x, y) = (\phi(t), \psi(t))$ with $\phi(t), \psi(t)$ algebraic functions; the endpoints A and B of the curve are given by $t = \alpha$ and $t = \beta$. Assume that the curve AB has no contacts and so has only intersections with the orbits. Starting on point M_1 with a semi-orbit, we may end up again on the curve AB in point M_1 which is the 'conséquent' of M_0. Nowadays we would call M_1 the Poincaré map of M_0 under the phase-flow of the ODE. Of course the semi-orbit may fail to return to AB, for instance because it will swirl around a focus far away or because it ends up at a node. It is also possible to choose the semi-orbit that moves in the opposite direction and return to the curve AB in M'; this point is called the 'antécédent' of M_0. If $M_0 = M_1$, the orbit is a cycle and Poincaré argues that returning maps correspond with either a cycle or a spiralling orbit. It is possible to discuss various possibilities with regards to the existence of cycles in which the presence or absence of singular points plays a part.

This analysis has important consequences for the theory of limit cycles. Semi-orbits will be a cycle, a semi-spiral not ending at a singular point, or a semi-orbit going to a singular point. Interior and exterior to a limit cycle there has always to be at least one focus or one node. Of the various possibilities considered it is natural to select annular domains, not containing singular points and bounded by cycles without contact. Such annular domains are often used to prove the existence of one or more limit cycles (Poincaré–Bendixson theory).

We note that Poincaré generalized the 'théorie des conséquents' later for higher-dimensional systems.

Non-integrability of Hamiltonian systems. In the eighteenth and nineteenth centuries, the integrability of conservative dynamics, later formulated as Lagrangian or Hamiltonian systems, was not in doubt among scientists. The only remaining problem, one thought, was to find the integrals in explicit physical models. In that period one could solve the gravitational two-body problem, one also made advances in the theories of rotating rigid bodies and rotating fluid masses and it seemed a matter of time and ingenuity to solve other cases like the gravitational three-body problem. In 1890, Poincaré, in his prize-winning essay [14] for the birthday of the Swedish king Oscar II, overturned this philosophy. Hamiltonian systems with two or more degrees of freedom are in general non-integrable; to put it differently, they are integrable only under additional assumptions. It took a very long time for scientists to understand and accept this result. Partly this was due to the philosophical impact of this non-existence theorem, but this can not be the full explanation as it was not simply a destructive result; it was accompanied by an extensive collection of new qualitative and quantitative concepts and tools to study dynamical systems. It is difficult to understand why so few scientists continued this line of research, perhaps it was the requirement of both analytic and geometric insight; people who followed it up in the first half of the twentieth century were Birkhoff, Denjoy, Siegel and Kolmogorov.

In [15], the fundamental theorem is formulated and proved in the case of the time-independent $2n$-dimensional Hamiltonian equations of motion

$$\dot{x} = \frac{\partial F}{\partial y}, \; \dot{y} = -\frac{\partial F}{\partial x}$$

with small parameter μ and the convergent expansion $F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$; F_0 depends on x only and the Jacobian is non-singular, $|\partial F_0/\partial x| \neq 0$. Suppose $F = F(x, y)$ is analytic and periodic in y in a domain D; $\Phi(x, y)$ is analytic in x, y in D, analytic in μ and periodic in y:

$$\Phi(x, y) = \Phi_0(x, y) + \mu\Phi_1(x, y) + \mu^2\Phi_2(x, y) + \cdots$$

The statement is then that with these assumptions, $\Phi(x, y)$ can not be an independent first integral of the Hamiltonian equations of motion unless we impose further conditions.

In the prize essay [14] and in the *Méthodes Nouvelles* [15] the approach to non-integrability is carried out in different ways. In the prize essay of 1890, the explicit theorem of non-integrability is formulated for two degrees of freedom systems with as application the planar, restricted, circular three-body problem. The starting point in the prize essay [14] is an unstable T-periodic solution with two zero characteristic exponents, one positive and one negative. The last two exponents correspond with an unstable and a stable invariant manifold emanating from the periodic solution; Poincaré calls these invariant manifolds 'surfaces asymptotiques'. Using series expansions for the solutions on the manifolds, he finds an infinite number of intersections instead of merging of the manifolds. This precludes the existence of homoclinic manifolds that would indicate the presence of a second integral. In the prize essay, the description of the geometry of the dynamics is tied in with the non-integrability results.

In the *Méthodes Nouvelles* [15], chapter 5 of volume 1, the technique is first analytic: a second integral should Poisson commute with and be independent of the Hamiltonian; expanding the second integral with respect to a suitable small parameter and applying these conditions leads to a contradiction unless additional assumptions are made (see also [21]). The dynamics and its geometry is later extensively studied in volume 3.

The proof is in both publications inspired by the actual Hamiltonian dynamics of stable and unstable manifolds. In chapter 32 of [15] we find the famous description of chaotic dynamical behaviour when considering the Poincaré section of an unstable periodic solution in a two degrees of freedom Hamiltonian system:

"If one tries to represent the figure formed by these two curves with an infinite number of intersections whereas each one corresponds with a double asymptotic solution, these intersections are forming a kind of lattice-work, a tissue, a network of infinite closely packed meshes. Each of the two curves must not cut itself but it must fold onto itself in a very complex way to be able to cut an infinite number of times through each mesh of the network.

One will be struck by the complexity of this picture that I do not even dare to sketch. Nothing is more appropriate to give us an idea of the intricateness of the three-body problem and in general all problems of dynamics where one has not a uniform integral and where the Bohlin series are divergent."

The double asymptotic solutions are the remaining homoclinic solutions that are produced by the intersections. The Bohlin series are formal series obtained by Bohlin for periodic solutions in celestial mechanics.

The Poincaré–Birkhoff theorem

A number of famous proofs developed by Poincaré to prove the existence and approximation of periodic solutions have been based on the implicit function theorem. In these cases one starts with a known periodic solution, for instance in the case of the three-body problem the limit case of one of the masses being zero, thus reducing the system to the known gravitational two-body problem. Starting with the known periodic solutions of this limiting problem one can try to obtain periodic solutions by continuation when increasing the vanishing mass from zero to a small value. Poincaré was bothered by the restriction of obtaining solutions in this way that are always close to known solutions, as the global view of the dynamics is still missing. He was able to formulate an important theorem regarding this question. In 1912, not long before his death, he wrote in the *Rendiconti del Circolo Mathematica di Palermo* [18]:

"I have never made public a work that is so unfinished; so I believe it is necessary to explain in a few words the reasons that have induced me to publish it and to start with the reasons that brought me to undertake this. Already a long time ago, I have shown the existence of periodic solutions of the three-body problem. However, the result is not quite satisfactory, for if the existence of each type of solution had been established for small values of the masses, one did not see what would happen for much larger values, which of the solutions would persist and in which order they would vanish. Thinking about this question I became convinced that the answer would depend on a certain geometric theorem being correct or false, a theorem of which the formulation is very simple, at least in the case of the restricted problem and of dynamics problems that have not more than two degrees of freedom."

George David Birkhoff (1884–1944)

Poincaré adds that during two years he had tried to prove the theorem but without success. However, he was absolutely convinced that the theorem was correct. What to do? Leave the matter rest?

"It seems that under these conditions, I would have to abstain of all publication of which I had not solved the problems. After all my fruitless efforts of long months, it seemed to me the wisest road to let the problem ripen and put it off my mind for a few years. That would have been very good if I had been certain that I could retake it some time, but at my age I could not say so. Also the importance of the matter is too great and the amount of results obtained already too considerable . . ."

He had already been suffering from serious prostate problems for several years and it seemed to be a waste to keep all these ideas

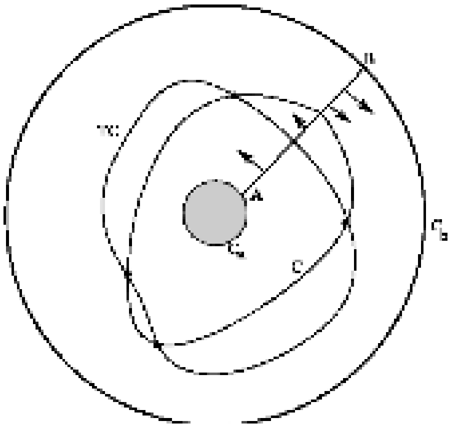

Figure 3 Twist map T of a ring shaped domain R

fruitless. As it turned out he was right, it is a beautiful fixed point theorem that combines geometric thinking with dynamics. In [12] it is classified under geometry, but it belongs as well to mechanics or differential equations. The theorem can be formulated as follows (see also Figure 3):

Poincaré–Birkhoff theorem. *Consider in* $\mathbb{R}^2$ *the ring* R *bounded by the smooth closed curves* C_a *and* C_b*. The map* $T : R \mapsto R$ *is continuous, one-to-one and area preserving. Applying* T *to* R *the points of* C_a *move in the negative sense, the points of* C_b *move in the positive sense (*T *is a 'twist' map). Then* T *has at least two fixed points.*

The theorem was proved by Birkhoff [2] (see also [21]) in a relatively simple way. It is difficult to understand why Poincaré did not produce a proof like this. Looking at the 39 pages of [18], one has the feeling that Poincaré just for once saw too many small difficulties, that he got bogged down in details.

The applications Poincaré had in mind can be indicated as follows. Consider a dynamical system derived from a time-independent Hamiltonian with two degrees of freedom. When studying the flow on a bounded energy manifold, one can make a Poincaré section of the flow in a neighbourhood of a periodic solution. This periodic solution is represented as a fixed point of the Poincaré map; if it is stable, the fixed point will be surrounded by closed curves corresponding with invariant tori around the periodic solution. The Poincaré map is area preserving, so that the application of the geometric theorem is possible if the twist condition has been satisfied. This can be checked by considering the rotation properties of the map on the closed curves.

An interesting aspect is that if one is able to apply the theorem for a Hamiltonian system, one finds not only two fixed points corresponding with two periodic solutions, but an infinite number. This is caused by the presence of an infinite number of the tori enabling us to construct an infinite number of rings. If the tori are close, the twist will usually be 'small' and in this case the period of the periodic solutions will be large. A strong twist of the map produces short-periodic solutions, a small twist long-periodic solutions.

Other applications can be found in problems of three-dimensional divergence-free flow and conservative billiard dynamics, see [3].

Conclusion

Henri Poincaré is sometimes described as 'the last universal scientist'. He stressed the importance of interaction between different fields of knowledge as a source of innovation in science. We restricted ourselves to the interaction of geometry and analysis in his work, for Poincaré there were many more fields of interaction from astronomy and astrophysics, physics, engineering and the humanities.

Acknowledgements

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Increasing insightful thinking in analytic geometry

Elsewhere in this issue Ferdinand Verhulst described the discussion of the interaction of analysis and geometry in the 19th century. In modern times such discussions come up again and again. As of 2014, synthetic geometry will not be part of the Dutch 'vwo – mathematics B' programme any more. Instead, the focus will be more on analytic geometry. Mark Timmer and Nellie Verhoef explored possibilities to connect the two disciplines in order to have students look at analytical exercises from a more synthetic point of view.

Although analytic geometry is a wonderful technique to prove a variety of theorems in Euclidean geometry in a convincing and easy manner, it rarely provides many insights. Secondary school students often apply it without any consideration of what they are actually doing. We conjecture that this leads to fragmented understanding. Rather than developing an overall picture of the geometric concepts the students are working with, the analytic and synthetic geometry remain isolated domains. This results in limited understanding of the mathematical structures at hand, and a limited set of techniques and strategies for solving exercises from these different domains. Analytic geometry becomes an end in itself; students manipulate formulas without any feeling for the underlying concepts.

Additionally, an analytical approach might sometimes even be much more cumbersome than a synthetic argument. By using analytical techniques for dealing with geometric figures, students sometimes forget about the properties of these objects, resulting in lengthy, unnecessary calculations.

In the context of the first author's Master's thesis for his mathematics teaching degree at the University of Twente, we tried to emphasise the underlying concepts of synthetic geometry when covering a chapter on analytic geometry. This was often accompanied by visualisations using the GeoGebra computer programme. The overall goal was to provide students a richer understanding of geometry [12]. More specifically, we were hoping for them to develop richer *cognitive units* [2]. That way, students understand better how different representations of geometric concepts such as ellipses relate, and are able to quickly switch between them. Hence, they might work more efficiently when solving exercises for which a purely analytical approach is unnecessarily difficult.

We already extensively discussed the lesson series and research project that resulted from the ideas above in a previous article [13]. Here, we elaborate more on the theoretical background regarding cognitive units and visualisation of geometric objects. Moreover, we discuss the way in which the results of this research project were put into practice as a workshop during the National Mathematics Days (NWD).

Underlying school mathematics

Our research primarily focused on the *ellipse*. This mathematical object can be defined as follows.

Definition 1. An ellipse is a set of points that all have the same sum of distances to two given focus points.

Definition 2. An ellipse is a set of points that are equidistant from a circle (the *directrix* circle) and a point within that circle.

The first definition is illustrated in Figure 1, the second one in Figure 2.

It is not hard to see that these two definitions coincide. In Figure 1, by definition $F_1P_1 + P_1F_2 = F_1P_2 + P_2F_2$; let this constant be r. In Figure 2, $MP_i + P_iF$ equals the radius of the circle, for both point P_1 and P_2, and all other points on the ellipse. The ellipse consisting of all points that are equidistant from a point F and a circle with centre M and radius r, therefore coincides with the ellipse consisting of all points with cumulative distance r to M and F. Stated differently, M and F are the focus points of the ellipse in Figure 2.

Placing an ellipse in a Cartesian coordinate system with the focus points on the horizontal axis (see Figure 3), we can show that it coincides with the set of points (x, y) such that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Here, a is half of the length of the horizontal axis, and b half of the length of the vertical axis. In Figure 3 this yields $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Interestingly, such an analytical representation relates in several ways to the

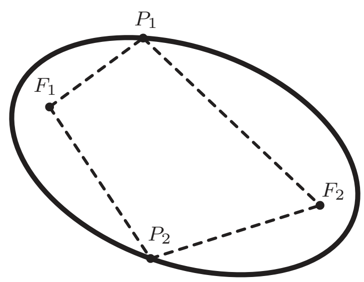

Figure 1 Equal cumulative distance to two focus points

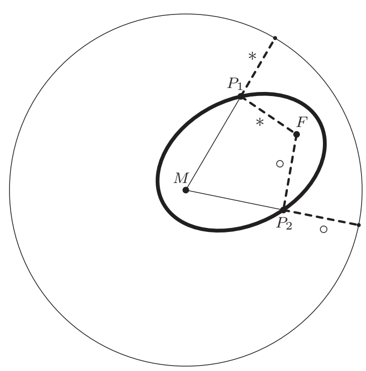

Figure 2 Equal distance to circle and point

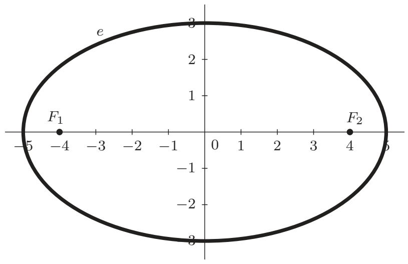

Figure 3 An ellipse in a coordinate system

synthetic definitions discussed above. For instance, $2a$ corresponds to the radius of the directrix circle, and $2\sqrt{a^2 - b^2}$ is the distance between the focus points.

We expected proficiency in such conversions between the analytical and the synthetic domain to increase understanding and insight, helping students solve exercises more effectively and efficiently.

Theoretical framework

In this study we investigated students' cognitive items with respect to geometric objects. In particular, we assessed the effects of a teaching method based on visualisation and synthetic geometry on these units. Hence, this section provides an overview of the theory regarding cognitive units and visualisation.

Cognitive units

The human brain is not capable of thinking about many things at once. Complicated activities such as mathematical thinking therefore have to be made manageable by abstracting away unnecessary details and focusing on the most important aspects [2]. The term *cognitive unit* originated from this idea:

"A cognitive unit *consists of a cognitive item that can be held in the focus of attention of an individual at one time, together with other ideas that can be immediately linked to it."* [10]

The 'cognitive item' mentioned here could be a formula such as $a^2 + b^2 = c^2$, a fact such as $10+3 = 13$ or a mental image of an ellipse. The connectivity between cognitive items and related ideas depends on the degree of understanding. For instance, most people would probably immediately relate $3+4$, $4+3$ and 7, and hence have strong connections between these cognitive items. They can then be considered as a single cognitive structure: a cognitive unit.

Barnard and Tall emphasise the importance of *rich* cognitive units, having strong internal connections between different objects or representations of objects, and leading to powerful ways of thinking. In our case, several different characterisations of the ellipse are considered. Initially, such characterisations will probably not be strongly connected in the students' brains. Later on, rich cognitive units might develop, allowing the students to perceive the characterisations as different representations of the same object. This is expected to yield more efficiency and understanding.

Compression to rich cognitive units. Rich cognitive units do not develop out of thin air. At first, a student will have a fragmented understanding of a new concept. Then, several different approaches might be needed to obtain a full understanding. However, once a concept has been fully understood, a significant *mental compression* can often be observed. Thurston explains how this results in a complete mental perspective — although at first obtained by a long process — to be easily used as part of a new mental process [11].

The notion of compression is applied on the one hand for the compression of knowledge into small cognitive items [3], and on the other hand for the way in which different cognitive items are coupled into strongly-connected cognitive units [10]. Since both processes yield richer cognitive units, we do not distinguish between these two meanings.

Causing compression. In order to induce compression, brain sections have to be connected to such an extent that addressing one of them also activates the others. After all, this makes the combined knowledge and understanding of these sections function together as a single cognitive structure [10].

More specifically, compression can be brought about in several different ways [9]. A student could categorise concepts or perform thought experiments, leading to connections between properties of those concepts. Repeatedly practising certain procedures until they are automated may also yield rich cognitive units. Finally, compression can be induced by *abstraction*: introducing symbols or names. Gray and Tall indeed indicate that we can only effectively talk about phenomena once they have been given a name [3]. As this compresses them to a cognitive unit, it enables us to think about them in a more sophisticated manner.

Visualisation

In this study, the underlying concepts from synthetic geometry were often visualised using GeoGebra, a computer programme for dynamic geometry [4]. The geometric objects under consideration indeed perfectly fit dynamic visualisation. For instance, we can easily use an equation for an ellipse and a slider determining its parameter a, to teach students this parameter's effect on the ellipse.

Scientific literature indicates that visualisation may improve mathematical understanding, although this does not necessary has to happen. Stols explains how the use of IT — more specifically, GeoGebra and Cabri 3D — only positively affects geometric insights of students that did not have much understanding yet, and even then only marginally [8]. He recommends to deploy applications such as GeoGebra to improve visualisation skills and conceptual understanding, and enable students to discover important relations. However, these programmes should not be expected to improve reasoning skills. We indeed only used GeoGebra for visualisation and to observe connections between concepts.

Langill also describes that software like GeoGebra should mainly be used as a supplement to non-technological sources, such as books [6]. She noticed that distance measuring and point dragging are among the most powerful applications of dynamic geometry. Therefore, we indeed combined visualisations with additional exercises, and extensively applied dragging and measurements to illustrate geometric properties.

Other researchers confirmed that technology can help students discover connections between different representations of the same concept, but also noticed that it should not be deployed too early [1]. They found that visualisations should be linked directly to knowledge that the students already possess, to avoid frustration and misconceptions. We therefore only used GeoGebra to clarify concepts the students were already familiar with, avoiding this pitfall.

Despite the potential merits of dynamic geometry software, it is still not used very often. Stols and Kriek report that a negative attitude towards the added value of such software, as well as a lack of confidence in their own technical skills, prohibit teachers from using applications like GeoGebra [14]. Zhao, Pugh, Sheldon and Byers also reached this conclusion, and observed that teachers have to take small evolutionary steps when introducing ICT in the classroom; a revolutionary approach would only lead to failure and frustration [15].

In this study, GeoGebra was only used by the teacher. Obviously, it is also possible to have the students play with the application. Although this is indeed expected to help students discover geometric theorems [7] or understand geometric transformations [5], we only applied GeoGebra for demonstrations. After all, we did not focus on developing new geometric skills, but more on the application of available geometric knowledge in the context of analytic geometry.

This study

We performed our study in a vwo 5 mathematics D class at the Stedelijk Lyceum Kottenpark in Enschede. Since this class consisted of only four students (for privacy reasons all addressed by 'he' in this article), we were able to observe the students in much detail and question them individually. The researcher taught Chapter 14 of the *Getal & Ruimte* vwo D4 method. This chapter covers symmetry, parametric equations and difference quotients, based on parabolas, ellipses and hyperbolas.

We tried to encourage the students to focus on connections between synthetic and analytic geometry in three different ways: (1) by giving additional explanations — often accompanied by GeoGebra visualisations — to make students aware of what they are doing, (2) by discussing how several analytical exercises from the book can be solved more easily using geometric reasoning, and (3) by introducing a number of new exercises for the students to practice these skills on. We refer to [12–13] for an extensive description of the lesson series.

Semi-structured interviews before and after the lesson series have shown quite a different effect on each of the four students. For one of them, the focus on synthetic geometry seemed to work out poorly. This student showed only limited knowledge and insight, both before and after the lesson series. He preferred to rely on an analytical approach, and already declared upfront to rather just calculate than think of a smarter way to solve an exercise. Additionally, he often indicated to not have much confidence in his own mathematical understanding, explaining his preference for structured rules and procedures.

The other three students were much more enthusiastic, and showed a positive attitude towards the new way of approaching analytic geometry. They most liked the feeling of deeper understanding, as well as the simplicity to achieve results. One student indeed showed considerably more insight during the post-test. He switched rapidly between different representations of the same concept, for instance by using symmetry for an analytical exercise and by combining both definitions of the ellipse in a smart manner. Additionally, he often first took a moment to think before relying on calculations, and showed growth in his associations with geometric concepts.

The other two students showed slightly less progress, but still improved visibly. They were able to identify more representations and more often applied geometric concepts such as symmetry. Interestingly, it appeared that some insights were present, but only surfaced after considerable encouragement. This indicates that certain connections between cognitive items have been made, but also that more practice is needed to enable fast switching between the accumulated knowledge from different domains.

National Mathematics Days

To share our findings with a larger group of teachers, we conducted a workshop during the most recent National Mathematics Days (www.fisme.science.uu.nl/nwd). There appeared to be quite some interest in our topic; teachers were happy to discuss a more insightful manner of working with analytic geometry.

After a short introduction of the subject, the teachers were asked to work on some of the exercises the students also tried to solve during their post-test. They intensely calculated and discussed, and appeared to pursue many different approaches. We found that they did not always fully use all available data and possible connections to other representations. The determination to solve the difficult exercises, however, was inspiring. Such an attitude would benefit every student!

The teachers asked many questions about the translation from our ideas to the classroom: how can we make students follow our approach, combining different representations and thinking before computing? As we mentioned before, frequent practice seems to be key. The workshop participants were pleased to hear and experience a creative way of addressing synthetic geometry in the current mathematics curriculum.

More details on the lessons and exercises can be found in [13]. For an extensive description of the research project, we refer to [12]. Both articles, as well as all material used at the NWD, can be found at http://fmt.cs.utwente.nl/~timmer/research.php.

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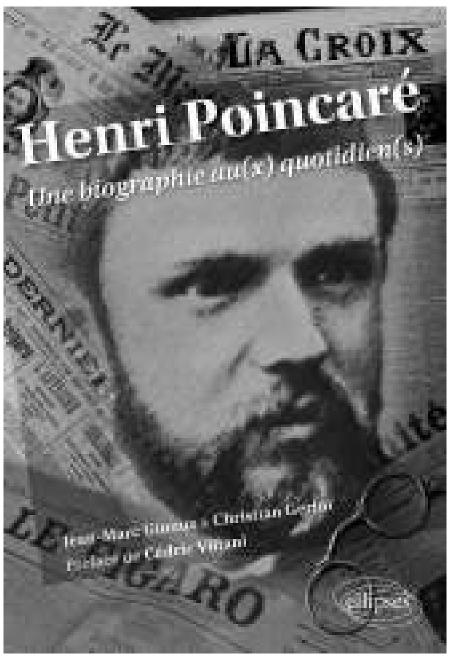
Book descriptions

New books about Henri Poincaré

In the 'centennial year' 2012 there will appear three books on life and work of Henri Poincaré. Each of the respective authors Jean-Marc Ginoux, Ferdinand Verhulst and Jeremy Gray gives a short description of his own book.

Jean-Marc Ginoux and Christian Gerini
Henri Poincaré: a biography through the daily papers
World Scientific, December 2012
Original French edition:
Henri Poincaré: une biographie au(x) quotidien(s)
Ellipses Marketing, July 2012
ISBN 9782729874070, 304 p., €24,-



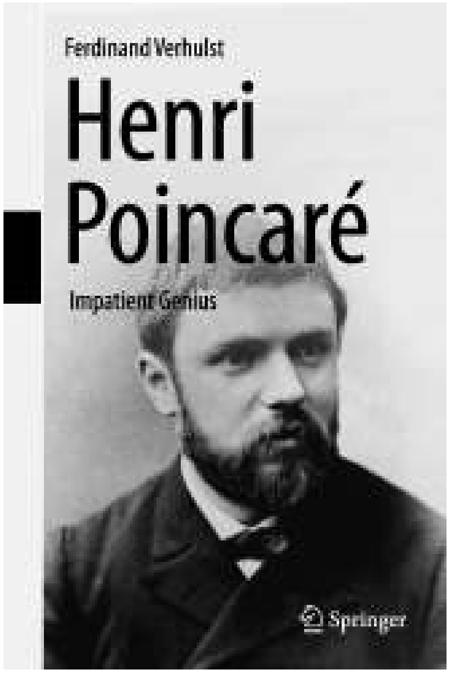
The original French edition

This book presents an original portrait of the famous French mathematician Henri Poincaré through from what the newspapers of its time said about him. An abundant choice of press cuttings (*The Washington Post*, *The New York Times*, *The Sun*, *The Times*, *The Herald Tribune*,...) allows to discover the most significant events of his career but also his role in the public space, both for its many scientific and technical skills as for his philosophical insights.

Moreover, this approach enables on the one hand to rebuild his biography highlighting many unknown anecdotes of his life (his first name was not Henri but Henry, he received his *baccalauréat* in science with zero in mathematics) and of his career (he was involved in a controversy about the rotation of the Earth from 1900 to his death, his intervention in the Dreyfus affair has played an important role) and on the other hand to evaluate the reception of the scientific and philosophical works of Henri Poincaré in the daily papers, i.e. by ordinary people. As an example, it may be interesting to discover how the newspapers reveal the 'mysteries' of the stability of the solar system when Poincaré won the King Oscar II Prize in 1889, or how they comment his election at the French Academy or his philosophical writings...

Contents

Ferdinand Verhulst
Henri Poincaré, impatient genius
Springer, August 2012
ISBN 9781461424062, 260 p., €39.95



This biography consists of two parts; first it describes the life and career of Henri Poincaré, the second part deals with scientific documents and details.

Working myself in the field of dynamical systems and chaos and taking part in the discussions on possible dramatic changes of the climate and other models for reality involving chaos, I became quite soon confronted with the work of Henri Poincaré. He was the first to notice the possibility of chaotic motion in classical mechanical systems, his insights and results were deep and far-reaching, using techniques that were understood only in the sixties of last century. He did these explorations around 1880, nearly 35 years old. Working in this field of dynamical systems, I started to read his papers and books on the dynamics of the solar system and also his philosophical essays.

The style of his papers and books struck me as unusual. They are written in a kind of discourse between author and reader. The most-used sentence is "this is not all", after which the author starts once more to discuss his topic, elucidating new aspects. I was not accustomed to this style of writing. Mathematics books, especially research books, are usually written deductively and in a kind of secret language which makes them difficult to understand.

Poincaré's essays are special in a different way. He formulates simple but fundamental questions. For instance, how is it possible that after weeks of looking without success for the solution of a mathematical problem, after a night of restless sleep, one awakes in the morning with the solution of the problem? This question and his own experience induces him to explore the part played by the unconscious which in a kind of independent search of possibilities leads to the solution.

The philosophical essays have been collected in four books with many of such basic questions. Is science not a construction based on our fantasies, has it any relation with reality? Or, what is mathematics and what is physics; is mathematics (or astronomy) useful, why are people anyway engaged in science? All this in crystal-clear prose with understandable reasoning, so that in the beginning of the twentieth century in France and elsewhere, these books started to play a considerable part in public discourse. For instance his first philosophical book, *Science et Méthode* (1902), sold in the French edition until 1915 more than 20.000 copies.

I felt that Henri Poincaré was a very special scientist and man and this motivated me to collect material for a scientific biography. I found that there existed some French, but pretty old biographies, there was no biography in English. After some time I ran into difficulties. Poincaré was one of the most important scientists of all times and one of the greatest of the last 150 years. There is nobody alive who is competent in all the topics he worked on and who has a general view of all his work. I started to feel dejected and was prepared to stop the project. I talked to a German colleague and friend about this who said: "Poincaré is too great for all of us, so this is not a reason to end the project." It did not solve my problem, but this sobering statement helped me and I continued.

When writing a biography, more often than not, a certain sympathy or bond between autobiographer and subject arises. This happened in my case rather quickly. The wide interest of young Henri in language and arts, his curiosity directed at the explanation of natural phenomena, all this agreed very well with my own experiences in youth. For example, Poincaré noted that the word 'chaleur' (English 'heat') was not well-chosen in the context of thermodynamics; it suggests a quantity instead of a dynamical state. He also noted that well-chosen terminology helps progress; the word 'flux' from fluid mechanics started to play a part in other fields like the theory of electricity. This sensitivity for the use of language appeals strongly to me.

In studying his life, I was also struck by his open-mindedness and honesty. For instance, the famous German mathematician Felix Klein sent him a note with a theorem and proof which Klein valued highly. The proof needed correction, but Poincaré's answer was worse:

"Thank you very much for your last note, which you were so kind to send me. The results that you mention do interest me, and I will tell you why. I found them already some time ago, but without publishing them, because I wanted to clear up the proof somewhat. That is why I would like to know yours when you, from your side, have clarified this."

Another example is when the retrial of captain Dreyfus took place; he was convicted on false evidence. Poincaré was chairman of the committee that had to consider the scientific value of the proofs at the trial. He wrote:

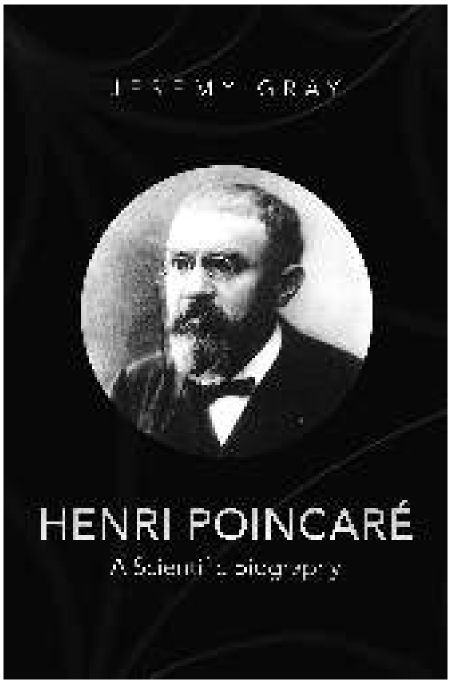
"There is nothing scientific in this evidence and I cannot understand your uneasiness. I do not know whether the accused will be found guilty, but if he is, it will be on other proofs. It is impossible that such an argumentation would be seriously considered by scientifically educated people without prejudice."

Note that Poincaré belonged to the Establishment, he was president of the Academy of Sciences, his cousin Raymond Poincaré was prime minister and later President of the French republic. In general, the French Establishment was strongly against Dreyfus.

There are many other examples like this. Henri Poincaré was an outstanding man and an eminent scientist.

Contents

Jeremy Gray
Henri Poincaré: a scientific biography
Princeton University Press, November 2012
ISBN 9780691152714, 616 p., $ 35,-



Henri Poincaré is chiefly remembered as one of the great mathematicians of all time, but he was also a leading physicist of his generation, nominated heavily if unsuccessfully for the Nobel Prize, and a prominent philosopher whose many essays are still in print a century later. The book is an attempt to survey entire output, and to bring out both its inventiveness and its surprising degree of coherence. It is also an attempt to situate him in his time and place, as an increasingly prominent member of French society, capable of intervening in numerous controversies both popular and technical, and actively involved in national and international scientific committees.

Poincaré was often drawn into the scientific controversies of his day. He advanced a philosophy of science that could cope with major changes in scientific claims, and defended them against opposition from Duhem and other Catholic apologists. He defended mathematics against what he saw as the self-contradictory attacks of logicians such as Russell and set theorists like Zermelo. His lectures on every branch of physics revitalised French higher-level education in the subject. Not only was he the leading French expert on the fast-moving theories of electricity, magnetism and optics, capable of original insights and penetrating criticisms of the ideas of Hertz and Lorentz, he was enthusiastically involved in the technological issues of his day, notably on the nature of radio waves. He served for many years on the Bureau des longitudes and was involved in their unsuccessful campaign to decimalise time as well as their geodetic survey of Peru. His position as the leading expert on probability made him an expert witness in the Dreyfus case that polarised the French society.

Numerous fields created or rewritten by Poincaré continue to be active fields of research. His novel interpretation of non-Euclidean geometry not only helped him create the flourishing topic of interactions between complex function theory (Fuchsian functions) and geometry, but also to formulate, and finally prove, the uniformization theorem for Riemann surfaces. It challenged contemporary ideas about the nature of space and drew him into controversies about the nature of knowledge. His presentation of topology began the modern study of the subject, recently highlighted by the successful resolution of the Poincaré conjecture. The modern theory of dynamical systems begins with his reformulation of celestial mechanics that led him to the discovery of chaotic motion even in simple physical systems. In his later work on the foundations of physics he came to propose the Lorentz group before Einstein, and in the last year of his life took up Planck's theory of quanta and was the first to propose that space and time are fundamentally atomic.

His popular essays often distilled his experience into analyses of how to think productively that are still fresh and surprising, and this is because they drew on one of his deepest concerns: What is it to understand something? Poincaré aimed not merely to advance knowledge and to obtain new results, but to organise his own thoughts so as to make his contributions most effectively. He advocated several methods for doing this, chief among them analogy and generalisation, and his test of his own understanding was to see how far it led him in the discovery of good new ideas. This emphasis on understanding, and not merely new results, partly accounts for his own successes and also helps to explain the lasting appeal of many of his essays. Mathematics and physics will always have new challenges, its latest discoveries will shed their lustre with age, and good new ideas can even turn out to be wrong, but what it is to understand mathematics and physics — what is involved in doing those subjects and doing them well — that will always be fresh. Moreover, it is clear that Poincaré took his own advice, and so these essays shed light on his own attitudes to science.

Topics for which Poincaré's work is perhaps less well-known today also bear out the unity of his thought. His earliest interest in number theory was the subject of his last paper, on the arithmetic significance of Fuchsian functions. His work on the partial differential equations of mathematical physics, and his enthusiastic acceptance of Fredholm's work, was immediately applied to his study of tides and ocean waves. His doctoral thesis, unpublished in his lifetime, connects to his work on the complex function theory of several variables and algebraic geometry, his early affinity for the work of Sophus Lie led him to contribute to major developments in the theory of Lie algebras.

Among his last popular lectures was his defence of the geometrical conventionalism that he had deployed in discussions of the possible non-Euclidean nature of space in the early 1890s against what he saw as a premature rush to embrace the ideas of Minkowski about space-time.

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Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 Euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 Euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is 1 December 2012.

Problem A (proposed by Mark Veraar)

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with values in $[0,+\infty)$ satisfying

$$\mathbf{P}(X_i > t) = \frac{1}{1+t}$$

for all i and all $t \geq 0$. Let $(c_n)_{n\geq 1}$ be a sequence of positive real numbers. Show that the sequence $(c_n X_n)_{n\geq 1}$ is bounded with probability 1 if and only if the series $\sum_{n=1}^{\infty} c_n$ converges.

Problem B (proposed by Simone Di Marino)

Determine all pairs (a, b) of positive integers such that there are only finitely many positive integers n for which n^2 divides $a^n + b^n$.

Problem C (proposed by Hendrik Lenstra)

Let $f \in \mathbb{Z}[X]$ be a monic polynomial, and let R be the ring $\mathbb{Z}[X]/(f)$. Let U be the set of all $u \in R$ satisfying $u^2 = 1$. Show that U has a ring structure with the following properties: the *zero element* is 1, the *identity element* is -1, the *sum* of two elements in U is their product in R, and the *product* $*$ in U is such that for all u, v, s, t in U the identity $u * v = s * t$ holds in U if and only if

$$(1-u)(1-v) = (1-s)(1-t)$$

holds in R.

Edition 2012-1 We received solutions from: Charles Delorme (Paris), Pieter de Groen (Brussels), Alex Heinis (Hoofddorp), Ruud Jeurissen (Nijmegen), Thijmen Krebs (Nootdorp), Paolo Perfetti (Rome), Merlijn Staps (Leusden), Roberto Tauraso (Rome), Sep Thijssen (Nijmegen), Rohith Varma (Chennai), Traian Viteam (Montevideo) and Hans Zantema (Eindhoven).

Problem 2012-1/A Let m and n be coprime positive integers. Let Γ be the graph that has the disjoint union $\mathbb{Z}/n\mathbb{Z} \sqcup \mathbb{Z}/m\mathbb{Z}$ as vertex set and that has for every $1 \leq i \leq m+n-1$ an edge connecting $i \pmod n$ and $i \pmod m$. Show that Γ is a tree.

Solution We received solutions from Charles Delorme, Pieter de Groen, Alex Heinis, Ruud Jeurissen, Thijmen Krebs, Merlijn Staps, Roberto Tauraso, Sep Thijssen, Rohith Varma, Traian Viteam and Hans Zantema. The book token goes to Hans Zantema. The following solution is based on that of Sep Thijssen.
Note that Γ has $m+n$ vertices and $m+n-1$ edges, so it suffices to show that Γ is connected. Without loss of generality we may assume $n \geq m$. Then note that for all $1 \leq i \leq m-1$, there is an edge connecting $i \pmod m$ to $i \pmod n$ and one connecting $i \pmod n$ and $i+n \pmod m$, since $i+n \leq m+n-1$. In particular, $i \pmod m$ and $i+n \pmod m$ are in the same connected component. Thus $n, 2n, \ldots, mn \pmod m$ lie in the same connected component. As n and m are coprime, this implies that $\mathbb{Z}/m\mathbb{Z}$ is inside a single connected component. Moreover, any vertex in $\mathbb{Z}/n\mathbb{Z}$ is connected to at least one of $\mathbb{Z}/m\mathbb{Z}$, hence Γ is connected.

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Oplossingen | Solutions

Problem 2012-1/B Is it possible to partition a non-empty open interval in closed intervals of positive length? Let Δ be a triangle (including its interior) and let $P \in \Delta$ be an interior point. Is it possible to partition $\Delta - \{P\}$ in closed line segments of positive length?

Solution *Solution to the former question.* We received solutions from Thijmen Krebs, Merlijn Staps and Sep Thijssen. The following is based on the submission of Sep Thijssen, who receives the book token.

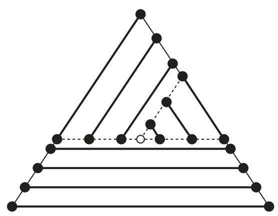
We will prove that it is not possible to partition a non-empty open interval (x_0, x_1) in closed intervals of positive length.

Let a partitioning $\mathcal{P}$ of (x_0, x_1) be given. We will now recursively define a sequence (x_n), starting with the given numbers x_0 and x_1. Consider the interval $[a, b]$ in $\mathcal{P}$ that contains $(x_n + x_{n+1})/2$. If n is even, we put $x_{n+2} = b$, and if n is odd, we put $x_{n+2} = a$.

The sequence $x_0, x_2, \ldots$ is increasing, and the sequence $x_1, x_3, \ldots$ is decreasing. Moreover, we have $|x_{n+1} - x_n| \leq (x_1 - x_0)/2^n$, and thus the sequence $(x_n)_n$ converges. Let $I \in \mathcal{P}$ contain the limit of $(x_n)_n$. Then I contains infinitely many of the x_n, which contradicts the fact that each x_n is an endpoint of the interval of $\mathcal{P}$ it lies in.

Solution to the latter question. We received solutions from Thijmen Krebs, Paulo Perfetti and Roberto Tauraso, Merlijn Staps, Sep Thijssen and Hans Zantema.

There are many ways to partition $\Delta - \{P\}$ in closed line segments of positive length; the following figure depicts one possibility.



Problem 2012-1/C A move on a pair (a, b) of integers consists of replacing it with either $(a+b, b)$ or $(a, a+b)$. Show that starting from any pair of coprime positive integers one can obtain a pair of squares in finitely many moves.

Solution This problem was solved by Charles Delorme, Alex Heinis, Thijmen Krebs and Hans Zantema. The book token goes to Charles Delorme. The following is based on the solution of Thijmen Krebs.

Without loss of generality we may assume that b is odd. Let p be a prime that is congruent to a modulo b and to 3 modulo 4. Such a prime exists by Dirichlet's theorem on primes in arithmetic progressions. By a finite number of moves we move from (a, b) to (p, b).

Similarly, choose a prime $q > b$ that is congruent to 3 modulo 4 and to b modulo p, and move to (p, q).

By quadratic reciprocity either p is a square modulo q or q is a square modulo p, but not both. Without loss of generality we assume that p is a square modulo q. So let x be an integer with $x^2 \equiv p$ modulo q. Let r_1 and r_2 be primes congruent to x modulo q with $r_1 \equiv 1$ and $r_2 \equiv 3$ modulo 4. Then by quadratic reciprocity, q is a square modulo either r_1 or r_2. In any case we find a prime r congruent to x modulo q such that q is a square modulo r.

Since $x^2 \equiv r^2$ modulo q, we can move to (r^2, q). As q is a square modulo r, it is also a square modulo r^2, so we may finally move to (r^2, t^2) for some t.