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Colofon

The Nieuw Archief voor Wiskunde is the quarterly journal of the Dutch Mathematical Society, the Koninklijk Wiskundig Genootschap. The journal is meant for anyone who is involved with mathematics professionally, as an academic or industrial researcher, student, teacher, journalist, or policymaker. Its goal is to report on developments in mathematics in general and in the Dutch mathematical community in particular.

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The Koninklijk Wiskundig Genootschap (KWG) is a national society for those who practise mathematics. Founded in 1778 with as its motto "untiring labour overcomes all", the KWG is the world's oldest national mathematics society. Members of the KWG receive the Nieuw Archief voor Wiskunde.

The contribution for members of the KWG is € 70 a year. Retired and unemployed members pay € 35. Students of Dutch universities and colleges and Ph.D. students can become member for € 25 a year. Students and those with a new Master of Arts or Master of Science degree can become member for one year for free. For members of the mathematics associations VVS, NVvW and NVORWO, the membership dues are reduced to € 50.

Members of *SIAM*, the *Société Mathématique de France*, the *Gesellschaft für Angewandte Mathematik und Mechanik*, the *Deutsche Mathematiker-Vereinigung*, and of the *American, Australian, Belgian, Indian, London, Spanish,* and *South-African Mathematical Societies* living outside of the Netherlands pay € 50 instead of the full membership fee of € 70 a year. Conversely, these foreign societies, as well as the VVS and the NVORWO, have reduced membership fees for KWG-members.

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Coming issues

issue	*publication date*	*contribution deadlines*
3	09–09–2008	–
4	08–12–2008	30–07–2008
1	08–03–2009	30–10–2008
2	08–06–2008	30–03–2009

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Universiteit Twente
Radboud Universiteit Nijmegen

On the cover

Rinus Roelofs, creator of mathematically inspired art, is one of the organizers of the conference Bridges in Leeuwarden (see page 155). The work on the cover is one of his series 'Connected holes'.

Amsterdam Archive

Welcome to a Dutch view of the world of mathematics. In response to the honour of having the Fifth European Congress of Mathematics in Amsterdam, the *Nieuw Archief voor Wiskunde* presents a special issue in English: the *Amsterdam Archive*. The Archive offers the usual sections, from opening *News* to the closing *Problem Section*, along with some unusual content. Try your skills on a selection of problems that have found no solutions over the past decade when posed in Dutch.

The editors are very pleased to have found a number of authors happy to prepare brief survey articles for the *Amsterdam Archive*. Please share our excitement over the latest developments in a range of fields from homology to history, and from logic to industrial mathematics. For your orientation into the Dutch world, we have included a number of specifically Dutch features, for example the prize-winning essay on the use of statistics in Dutch forensic research, which has been translated for the occasion. In addition, a laureate of many prizes gives his account of the optimization of Dutch train schedules and links it to a remarkable Russian optimization…or was it really Russian?

There is nothing more Dutch in mathematics than our own word for it, *wiskunde*, which was coined by Simon Stevin. Read how the society that is offering you this journal came to call itself *Wiskundig Genootschap* (and now *Koninklijk Wiskundig Genootschap*). As a welcoming contribution, there is a compilation of sights and museums (some as old as our very own Genootschap) that you do not want to miss while in the Netherlands. Even if the Escher Museum has been listed, the latest news is that Escher is *out*. As you may judge from the interviews in this issue, one can be a mathematician and play with Escher's work, while at the same time not particularly liking him as an artist. In contrast, there are artists that we do now particularly like (see the cover of this edition).

The most remarkable developments in mathematics in the Netherlands over the past few years are only obliquely reported in this issue, i.e. the signs of strengthening ties with society. It is good to see that research is now more firmly embedded, that students are returning to mathematics and that public awareness is growing.

In 2003, mathematical research in the Netherlands was reorganized on the basis of a strategy of clustering designed by a national committee for mathematics. In May of this year (2008), the successor committee presented its strategy for a new generation of mathematical research: *Concentration and Dynamics*. What looked like a painful operation, answering to diminishing support, five years ago has now been followed up by a refreshingly self-conscious approach. An essential part of our legitimacy is given by education. We have been asking ourselves, with good reason, where all the students have gone. Finally this year, the initiatives to address this issue promise to bear fruit and lead to substantially better figures of enrolment. Perhaps it has been helpful, too, that on the horizon signs of consensus are appearing on the role and style of mathematics in secondary education.

Little explicit proof of the changes in the societal embedding of research and education appears in this special issue. What is present is the improving communication. The incoming president of the Royal Academy of Sciences reveals the kissing factor in mathematics. However, when it comes to public awareness of mathematics in the Netherlands, few people have more effectively communicated than the Math Girls; read how they rule and follow the link to their prize-winning website.

The epitome of the good spirits of 2008, which I hold to be the most important of the recent development in mathematics in the Netherlands, is clearly the arrival of the Fifth European Congress of Mathematics to Amsterdam.

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The **Amsterdam Archive** is a special issue of the *Nieuw Archief voor Wiskunde*, 5/9–2, June 2008, issued on the occasion of the fifth European Congress of Mathematics, Amsterdam, July 13–18, 2008, edited by

Gerard Alberts, Editor-in-Chief,
Gunther Cornelissen, Patrick Oonincx, Martin Raussen, editors,
Chris Nunn, language editor.

Calendar
| Upcoming Events

July 2008

3 – 5 July
❑ **LOFT 2008**
Eighth Conference on Logic and the Foundations of Game and Decision Theory
location *Amsterdam*
info *www.illc.uva.nl/LOFT2008*

6 – 13 July
❑ **ICME11**
Eleventh edition of the International Congress on Mathematical Education
location *Monterrey, Mexico*
info *www.icme11.org.mx/icme11*

7 – 11 July
❑ **The Geometric Langlands Program**
Workshop organised by the Dutch research cluster Geometry and Quantum Theory as a side activity to the Fifth European Congress of Mathematics
location *Lorentz Center, Leiden*
info *lc.leidenuniv.nl*

7 – 11 July
❑ **International Workshop on Statistical Modelling**
Twenty-third international workshop
location *Universiteit Utrecht*
info *www.fss.uu.nl/iwsm2008*

10 – 22 July
❑ **49th International Mathematics Olympiad**
The 2008 edition of the World Championship Mathematics Competition for high school students takes place in Madrid.
location *Madrid, Spain*
info *www.imo-official.org*

13 – 18 July
❑ **Fifth European Congress of Mathematics**
European mathematical congress under the auspices of the European Mathematical Society and under patronage of the Koninklijk Wiskundig Genootschap
location *RAI, Amsterdam*
info *www.5ecm.nl*

18 – 23 July
❑ **The Netherlands Workshop on Graphs and Matroids**
Small workshop focussed on graphs and matroids, organized in conjunction with the 5ECM Minisymposium on Graphs and Matroids
location *Sittard*
info *homepages.cwi.nl/˜bgerards/workshop.html*

21 – 25 July
❑ **Operator Structures and Dynamical Systems**
Workshop as side activity of the Fifth European Congress of Mathematics about the relation between dynamical systems and the attached operator structures
location *Lorentz Center, Leiden*
info *lc.leidenuniv.nl*

24 – 28 July
❑ **Bridges Leeuwarden**
Annual conference on the connections between mathematics, music, art, architecture and culture. Escher birth year celebration
location *Leeuwarden*
info *www.bridgesmathart.org*

29 July
❑ **Family Math/Art day**
Mathematical Art Public Activity Day organised by the Bridges Conference
location *Leeuwarden*
info *www.bridgesmathart.org*

August 2008

11 – 15 August
❑ **Decoherence in Quantum Information Science**
Workshop on models of quantum decoherence
location *Lorentz Center, Leiden*
info *lc.leidenuniv.nl*

18 – 22 August
❑ **European Study Group with Industry**
66th European Studygroup with Industry, where mathematicians and industry interact
location *Lyngby, Denmark*
info *www2.mat.dtu.dk/ESGI/66*

Please submit items for this calendar to the following address.

Nieuw Archief voor Wiskunde,
Mathematical Institute, University Leiden,
P.O.Box 9512, 2300 RA Leiden.
Email: agenda@nieuwarchief.nl

18 – 23 August
❑ **USSE 2008**
Utrecht Summer School in Science and Mathematics Education
location *Universiteit Utrecht*
info *www.fi.uu.nl/nl*

22 – 23 August
❑ **Vakantiecursus 2008**
Holiday course for mathematics teachers. The same course will be taught at the CWI on 29 and 30 August
location *Technische Universiteit Eindhoven*
info *www.cwi.nl/events/2008/VC2008*

25 – 29 August
❑ **Dynamics Days Europe 2008**
Conference with lectures on non-linear dynamics that appear in climate dynamics, dynamics of biosystems and Hamiltonian systems
location *Technische Universiteit Delft*
info *dd2008.ewi.tudelft.nl*

25 August – 5 September
❑ **Dynamical heterogeneities in glasses, colloids and granular media**
Workshop for theorists and experimentalists
location *Lorentz Center, Leiden*
info *lc.leidenuniv.nl*

27 – 28 August
❑ **10th anniversary Eurandom**
Conference in honour of the tenth anniversary of Eurandom
location *Eurandom, Eindhoven*
info *www.eurandom.tue.nl*

27 – 28 August
❑ **Symposium Number Theory & Discrete Mathematics**
Two-day sympoisum in honour of the 65th birthday of Rob Tijdeman
location *Leiden*
info *www.math.leidenuniv.nl/˜smeets/tijdeman.html*

29 August
❑ **Seminar Toekomstvisie Wiskunde Nederland**
Seminar in honour of the 65th birthday of Rob Tijdeman about the vision for the future of mathematics in The Netherlands, followed by the farewell lecture of Rob Tijdeman
location *Leiden*
info *see info link above*

29 – 30 August
❑ **Vakantiecursus 2008**
Holiday course for mathematics teachers. The same course is taught at Technische Universiteit Eindhoven on 22 and 23 August
location *CWI, Amsterdam*
info *www.cwi.nl/events/2008/VC2008*

September–December 2008

22 – 26 September
❑ **Logic and information security**
Interdisciplinary workshop on combinatorial mathematics for bit exchange, information-based protocol analysis, zero-knowledge protocols and model checking
location *Lorentz Center, Leiden*
info *lc.leidenuniv.nl*

23 September
❑ **NWO-Talent Class**
One-day workshop for Ph.D. students and postdocs on the different facets of life as a scientist
location *NWO, Den Haag*
info *www.nwo.nl/talentendag*

5 – 12 October
❑ **Differential Equations, Function Spaces, Approximation Theory**
International conference dedicated to the 100th anniversy of the birthday of Sergei L. Sobolev
location *Novosibirsk, Russia*
info *math.nsc.ru/conference/sobolev100/english*

6 – 8 October
❑ **Young European statisticians Workshop-YES II**
Workshop for young statisticians
location *Eurandom, Eindhoven*
info *www.eurandom.tue.nl*

8 – 10 October
❑ **Woudschoten Conference**
33th Woudschouten Conference of the Dutch and Flemish Numerical Mathematics Societies. Special topics: numerically solving high-dimensional problems and bio-mathematics
location *Conferentiecentrum Woudschoten, Zeist*
info *wsc.project.cwi.nl/woudschotenN.php*

4 November
❑ **NWO-Talent Class**
One-day workshop for Ph.D. students and postdocs on the different facets of life as a scientist
location *NWO, Den Haag*
info *www.nwo.nl/talentendag*

13 – 14 November
❑ **Farewell address and workshop in honor of Geert Jan Olsder**
Workshop with lecturers Melikyan, Ho, Başar, Haurie, Heidergott, Cohen, Baccelli, Hautus, Bernhard, Filar, Pourtallier and Van der Woude
location *Technische Universiteit Delft*
info *www.tudelft.nl*

24 – 26 November
❑ **Locally adaptive filters in signal and image processing**
Third in a series of annual workshops at Eurandom devoted to signal and image analysis
location *Eurandom, Eindhoven*
info *www.eurandom.tue.nl*

1 – 5 December
❑ **KAM Theory and its applications**
Workshop on Kolmogorov-Arnold-Moser theory in dynamical systems theory
location *Lorentz Center, Leiden*
info *lc.leidenuniv.nl*

2009–2010

10 – 22 July 2009
❑ **50th International Mathematics Olympiad**
The 2009 edition of the World Championship Mathematics Competition for High School students
location *Bremen, Germany*
info *www.imo2009.de*

19 – 27 August 2010
❑ **International Congress of Mathematicians 2010**
Proposals concerning the scientific programme are welcomed by the Chair of the Program Committee, Hendrik Lenstra
location *Hyderabad, India*
info *www.mathunion.org/ICM/PC/2010.html*

News

| Mathematical Activities

Abel prize 2008 for Thompson and Tits

John Griggs Thompson of the *University of Florida* and Jacques Tits of the *Collège de France* have won the 2008 Abel Prize "for their profound achievements in algebra and especially for their shaping of modern group theory".

The Abel Committee of the *Norwegian Academy of Science and Letters* states that "Thompson revolutionised the theory of finite groups by proving extraordinarily deep theorems that laid the foundation for the complete classification of finite simple groups, one of the greatest achievements of twentieth century mathematics". Furthermore "Tits created a new and highly influential vision of groups as geometric objects. He introduced what is now known as a Tits building, which encodes in geometric terms the algebraic structure of linear groups." *www.abelprisen.no*

photo: University of Florida and Jean-François Dars/CNRS Images

2008 Abel Laureates John Griggs Thompson and Jacques Tits

Amsterdam student wins in Ostrava

Mathematics student Demeter Kiss of the *Vrije Universiteit* in Amsterdam has won the *Vojtech Jarnik* mathematics competition in Ostrava in the Czech Republic. In category II, the category for students who have finished their first two years at university, Kiss stayed three points ahead of the runner up. Another student from Amsterdam, Attila Herczegh, finished 9th in the same category, which saw a total of 66 contestants. This was the first year the *Vrije Universiteit* participated in this competition. *jarnik.osu.cz and www.scienceguide.nl*

Liebe Maria

In early 2006 mathematics, physics, astronomy and computer science students of the student association *De Leidsche Flesch* surprised friends and foes by offering a letter to then-minister of Education Maria van der Hoeven complaining about the level of their high school mathematics education (*Nieuw Archief voor Wiskunde*, March 2006). It made quite a stir back then in the Netherlands, but as it turns out also hasn't gone unnoticed abroad.

Aloys Krieg, Ferdinand Verhulst and Sebastian Walcher wrote an article for the *Deutsche Mathematiker Vereinigung* (German Association of Mathematicians) about the Dutch educational situation and the student initiative that got known under the name of the accompanying website *Lieve Maria* (Dear Maria).

Krieg and Walcher, both from the *Rheinisch-Westfälische Technische Hochschule Aachen*, hope that the article, which at the time of writing

This section offers an account of mathematical activities in the Netherlands. Future activities are announced and past activities are reported on. Would you like to see your announcement or report in this section? Send your contribution (± 350 words, if possible with illustration) to news@nieuwarchief.nl.
The news editor reserves the right to shorten or refuse contributions.

Editor: Yves van Gennip

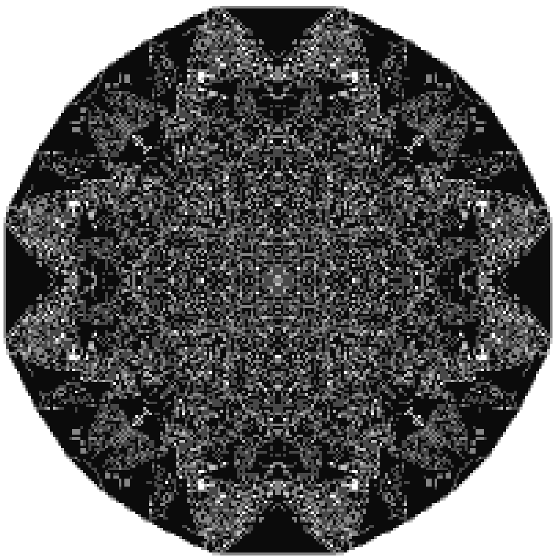
is still to appear in the *Mitteilungen der DMV*, will stimulate discussion within the *Deutsche Mathematiker Vereinigung* about mathematics education and possible changes therein in Germany. *Yves van Gennip*

Does mathematics run into the sand?

Not only kids like to play in sandpiles, mathematicians also find many exciting things going on between the silica grains. Former mathematics teacher turned researcher Anne Fey-den Boer received her Ph.D. degree for her research on models of these sandpiles. She found that they exhibit various forms of self-organisation and patterns are formed that are stable over the course of time. Fractal patterns can also develop.

The model Fey uses is simple to explain. The pile is modelled as a grid where each grid point has a height. With the passing of time the height of a grid point increases until it passes a critical value after which the excess mass has to be divided over neighbouring grid points. However, the behaviour of this model shows very complex and wide-ranging behaviour.

Models like this are also used to describe the movement of the Earth's crust, stock market fluctuations and the formation of traffic jams and so a better understanding of their behaviour is relevant to many other fields of science and applications. Here mathematics does not run into the sand. *Bron: www.nwo.nl*



Dispersion pattern of 366.600 particles from the origin

Math students wins ECHO Award

University of Amsterdam mathematics student Apo Cihangir has won the *ECHO Bèta Techiek Award*, a prize from the Dutch expertise center for diversity policy *ECHO* and the science and technology organisation *Platform Béta Techniek* for talented foreign, non-western, students. He won the prize because of his study results, organisational talent and social involvement. He is involved with the *IMC Weekendschool*, which provides additional education for young teenagers from poor neighbourhoods. *Bron: www.science.uva.nl*

Train of thought

Was the train you took to get from Schiphol airport to the 5ECM conference delayed? There is a good chance it was not, thanks to some mathematically inspired traffic management.

Andrea D'Ariano received a Ph.D. degree from the *Technische Universiteit Delft* last April for his research on automated systems for advising rail traffic managers. This research was fuelled by the wish of the *Nederlandse Spoorwegen* (Dutch Railways) to improve the punctuality its trains by streamlining the sequence of events during breakdowns and delays. Based on mathematical models and algorithms D'Ariano developed ROMA (Railway traffic Optimization Means of Alternative graphs), an automated system which can help rail traffic managers in making their daily decisions. Simulated studies showed that the use of ROMA led to less and shorter delays.

One drawback of ROMA is that detailed up-to-date traffic information must be available at all times, a demand that is not realistic at the moment, but which may become more feasible when trains start carrying their own GPS devices.

The research fits within the current trend of dynamical traffic management, which is applied on a small scale already at, for example, Schiphol station where it is decided only at the very last moment on which platform a train will enter the station. Have an undelayed trip home. *www.tudelft.nl*

Award for networks

Johan van Leeuwaarden of the *Technische Universiteit Eindhoven* has won the *Doctoral Dissertation award for Operations Research in Telecommunications* for his research into mathematical models for dealing with the ever increasing information flow in networks, like the internet.

His dissertation *Queueing models for cable access networks* dates back to 2005; this year it was awarded the aforementioned award during the biennial INFORM-conference. *Bron: www.tue.nl/cursor*

Study Group Mathematics with Industry 2008

On an early morning, Monday 28 January, about 80 mathematicians gathered at the University of Twente for a week of mathematical problem solving. First, Rector Magnificus Henk Zijm described each of the participating companies. This didn't cause him too many problems, as he has worked for five of them and is paying taxes to the sixth. He put a great emphasis on the importance of mathematical modeling. The opening was followed by a talk from STW representative Rik Janssen.

By then most participants were eager to learn about the six problems they would be working on during the week. First up was Yvo Boers, from Thales, who told us about the problem of placing non-perfect sensors in a domain in various situations. After a large number of questions and a well-deserved coffee break, Jeroen van der Scheer from the Water Board Regge and Dinkel explained the problems they face with the current climate change. Leo Kroon, from Dutch Railways NS, presented the next problem (how to park trains overnight) using many pictures and a computer game he had solved on the train that very morning.

After the excellent lunch Peter Steeneken and Jiri Stulemeijer explained the modelling wishes NXP had with respect to Micro Electro-Mechanical Switches. Then Kevin Dolan, representing Philips Medical Systems and AMC, left us all feeling slightly uncomfortable with his stories about needles being put into people's brains. I must admit, though, that the lively description of a dying neuron by his colleague Lo Bour helped achieve that feeling. Andreas ten Cate, Corus, fortunately presented a less invasive problem: how to efficiently model the changing composition of liquids during the solidification of molten mixtures. During the tea break various groups were formed. Some people immediately knew what problem they wanted to work on, some found it harder to make up their minds, while others kept circulating throughout the week (although, we were assured, only very few attendants had this privilege).

During this week, the mathematicians have interpreted the problems, then deformed and simplified them, carried out experiments, and have come up with promising results and many starting points for future research. I was impressed by what can be achieved in just one week, and I am looking forward to the next edition, 26-30 January 2009 in Wageningen. I have no doubt that it will be as great as this one.

Daily reports, details of the problems, and the various occurrences of the Study Week in the national Press can be found at the SWI website: wwwhome.math.utwente.nl/˜swi2008. SWI 2008 was sponsored by STW, NWO and also CWI, UT, CTIT and IMPACT. *Dan Roozemond*

photo: wwwhome.math.utwente.nl/ swi2008

Mathematicians deciding how best to help Corus model the aluminium casting process

Trained for efficiency

The *Nederlandse Spoorwegen* (Dutch Railways) have won the 2008 *Franz Edelman Award for Achievement in Operations Research and the Management Sciences*, which is presented annually by the *Institute for Operations Research and the Management Sciences* (INFORMS) to recognise “outstanding examples of operations research-based projects that have transformed companies, entire industries, and people’s lives”.

Fellow finalists are the Federal Aviation Administration, StatoilHydro, the City of Stockholm, the U.S. Environmental Protection Agency and Xerox. They were beaten by the Dutch entry *The New Dutch Timetable: The O.R. Revolution.*

In 2006 the Dutch passenger railway network had a traffic volume of 15.4 billion (short scale) passenger kilometers, which was a significant increase from the 8.0 billion in 1970. The timetable had never been structurally altered in all those years however and more and more scheduling problems began to arise.

Operations researchers constructed four solvers (*CADANS, STATIONS, ROSA* and *TURNI*) to construct a new cyclical timetable that can be repeated every hour.

The new schedule was introduced in December 2006. The year 2007 had an all-time passenger record, with a 10–15% increase in passenger demand on the lines with the largest timetable improvements. The percentage of trains arriving within three minutes of the scheduled time increased from 84.8% in 2006 to 87.0% in 2007, which is again an all-time high. Customer surveys show a radical improvement in the public’s perception of the railways. The more efficient resource schedules and the increased number of passengers have already resulted in an annual additional profit of 40 million Euros. The new timetable also made a further increase in railway transport on the current network possible.

It is not easy to localise the role played by the new timetable in all these improvements, but one result is directly visible: the *Edelman Prize* that was awarded last April. *Bron: www.informs.org*

A world of mathematics

Employees and students of the *Technische Universiteit Eindhoven* have developed *Wiskunde Wereld* (Mathematics World), an activity for third year high school students in which they come into contact with eight different mathematical subjects. The goals is to give students a better idea of what mathematics is and how it is applicable to the world around us by letting them play mathematical games. *w3.win.tue.nl*

Petition for Ibni Oumar Mahamat Saleh

Chadian mathematician and polician Ibni Oumar Mahamat Saleh was arrested by members of the presidential guard on February 3. At the time of writing no confirmed news about his whereabouts or condition had been released. The Mathematical Societies of France *SMF*, *SMAI*, and *SFdS* initiated a ‘petition of the international mathematical community’ to be delivered to the presidents of the Republic of Chad and the French Republic asking for the truth about Ibni Oumar Mahamat Saleh to be disclosed.

Ibni Oumar Mahamat Saleh is Secretary-General of the opposition party *PLD*. People fear for his life after fellow opposition leader Ngarlejy Yorongar, who was imprisoned along with him but was released, told that Ibni Oumar Mahamat Saleh was severely beaten and had probably died. *smf.emath.fr/en/PetitionSaleh*

Wolf prizes for Deligne, Griffiths and Mumford

The 2008 Wolf Foundation Prize in mathematics has been jointly awarded to Pierre R. Deligne, Phillip A. Griffiths and David B. Mumford for their pivotal work in algebraic geometry.

Deligne was awarded the prize “for his work on mixed Hodge theory; the Weil conjectures; the Riemann-Hilbert correspondence; and for his contributions to arithmetic”, Griffiths “for his work on variations of Hodge structures; the theory of periods of abelian integrals; and for his contributions to complex differential geometry”, and Mumford “for his work on algebraic surfaces; on geometric invariant theory; and for laying the foundations of the modern algebraic theory of moduli of curves and theta functions”. *www.wolffund.org.il*

Pierre Deligne, Phillip Griffiths and David Mumford, winners of the 2008 Wolf Prize (photos: Deligne Mumford: www.wolffund.org.il; GGriffiths: Cliff Moore)

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Mathematical connections

It is truly a golden age for mathematics. We are witnessing an unprecedented confluence of fields that demonstrates the unity of mathematical thought. Eugene Wigner has eloquently named this phenomenon 'the unreasonable effectiveness of mathematics'. Mathematical ideas have the remarkable ability of turning up in the most diverse contexts. Paraphrasing the opening of Wigner's essay, whenever one meets a π, a sine or a cosine, one has to ask: where is the circle hidden in my problem?

This deep connectivity is often overlooked by the general public who easily get intimidated or frightened by mathematics. It is a conventional wisdom among publishers of popular science books that every equation cuts the potential readership in half. Most people therefore overlook a modest but crucial ingredient in these equations: the equals sign. In its archetypal form $A = B$, the equals sign connects two worlds represented by A and B. Through it ideas can flow from A to B and back, as if the equals sign conducts the electric current that lights up the 'Aha!' light bulb in the mind indicating the insight gained. Albert Einstein was an absolute master in finding equations with that property. Take $E = mc^2$, which connects mass and energy and is, without a doubt, the most famous equation in the public imagination. The equations of general relativity, although less catchy and well-known, link in an equally surprising and elegant way the worlds of geometry and matter. Another beautiful illustration is given by the Index Theorem of Atiyah and Singer, where the left hand side represents the world of global analysis and partial differential equations, and the right hand side represents the world of geometry and topology. In that sense the discovery of a great theorem can be the mathematical equivalent of the collision of two continents. Just as North and South America bumped into each other three million years ago, and animals and plants started moving up and down along the Panama Isthmus, mathematical ideas can flow between fields if they are connected by a great theorem, generating new forms and finding new applications.

If we pursue the analogy between continental drift and the movement of scientific fields then we are living in a time of exceptional geological activity. Interdisciplinary fields are developing at an unprecedented speed. My own discipline of mathematical physics finds itself in a particularly active period. Recent events illustrate this. The 2006 Fields Medallists Okounkov, Perelman and Werner all used crucial ideas from physics. The 2008 Crafoord Prize was awarded to Fields Medallists Kontsevich and Witten for their work connecting geometry and quantum physics. The new Simons Centre for Geometry and Physics at Stony Brook University, generously endowed by mathematician and financier James Simons, is especially directed toward this active area of research.

The synergy between physics and mathematics is definitely not a new phenomenon. Mathematics has a long history of drawing inspiration from the physical sciences, going back to astrology, architecture and land measurements in Babylonian and Egyptian times. Certainly this reached a high point in the scientific revolution of the 17th century with the development of calculus and classical mechanics. One of its leading architects Galileo has given us the famous image of the 'Book of Nature', which is waiting to be decoded by scientists. In *Il Saggiatore* he writes: 'Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these one is wandering in a dark labyrinth'. And this deep respect for mathematics didn't disappear after the 17th century. At the beginning of the last century we again saw a wonderful intellectual union of physics and mathematics when the great theories of general relativity and quantum mechanics were developed. This was closely watched throughout the mathematical world; mathematicians actively participated, in particular in Göttingen where Hilbert, Minkowksi, Weyl, Von Neumann and many others made important contributions to physics.

Theoretical physics has always been fascinated by the mathematical beauty of its equations. Here we can even quote Feynman, who was certainly not known as a connoisseur of abstract mathematics: 'To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in'. (Of course, he has also stated: 'If all mathematics disappeared today, physics would be set back exactly one week,' to which a mathematician had the clever answer: 'This was the week that God created the world'.)

But despite the warm feelings expressed by Feynman, the paths of fundamental physics and mathematics started to diverge dramatically in the 1950s and 1960s. In the struggle to understand the stream of new subatomic particles, physicists were close to giving up the hope of an underlying mathematical structure of nature. On the other hand mathematicians were very much in an introspective mode in that period. Dyson stated in his Gibbs Lecture in 1972: 'I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce'. But in some sense these were famous last words because at that time the Standard Model of elementary particle physics was born. All the building blocks of that model — gauge fields, curvatures, bundles, covariant derivatives, spinors, Dirac equations — have turned out to have completely natural mathematical interpretations. Soon mathematicians and physicists started to build this dictionary and through the work of Atiyah, Singer, Chern, Yang, 't Hooft, Polyakov and many, many others a new period of fruitful interaction between mathematics and physics was born.

These days, under the influence of quantum theory, the collision of physics and mathematics is producing many new fertile lands. This is particularly true for string theory. Its stimulating influence in mathematics will have a lasting and rewarding impact, whatever its final role in fundamental physics turns out to be. The number of mathematical subfields that come together is dizzying: analysis, geometry, algebra, topology, Lie theory, combinatorics, probability, operator algebras, K-theory, categories — the list goes on and on. One starts to feel sorry for the students who have to learn all this! And it is not only the case that physicists are using advanced mathematical techniques. There are many ideas coming out of physics that have truly influenced pure mathematics. In this respect I like to paraphrase Wigner and speak of 'the unnatural effectiveness of physics in modern mathematics'. To mention just three areas: representation theory, low dimensional topology and algebraic geometry have all been transformed in the last few decades.

In retrospect this development is not as surprising as one might think. A traditional role of physics has been to provide a 'natural' context for mathematical concepts. This has been the case for the classical world but is even more so for the quantum world, which represents a more fundamental description of reality. Even though our intuition, based on everyday experience, is not very well-suited to understanding the properties of elementary particles, ideas from quantum field theory carry tremendous mathematical power. For example, whereas in classical mechanics a particle travels from A to B along a fixed path determined by a variational principle (for example, a geodesic), in quantum mechanics one should consider the 'sum over all histories'; all paths from A to B have to be considered in a weighted ensemble with the classical trajectory being the most likely. Therefore quantum theory naturally leads to summations over objects in certain sets.

A striking example of this principle is mirror symmetry. String theory suggests computations of the number of complex curves on a manifold not degree by degree but in a single stroke by relating the counting function to a classical object (the period of a holomorphic form) on the associated 'mirror manifold', which can have a very different topology. A famous example of the power of mirror symmetry is the original computation of the quintic Calabi-Yau hypersurface in projective four-space by Candelas and collaborators (*Nucl. Phys.* **B359** (1991) 21). This is a hard problem. The 2875 lines (degree one curves) on the quintic are a classical result from the 19th century. The 609250 different conics were only computed around 1980. Finally, the number of twisted cubics 317206375 was the result of a complicated computer program. However, thanks to string theory, we now know the full expansion. Here are the next terms: 242467530000, 229305999987625, 248249742118022000, 295091050570845659250, etc. When considered as coefficients in a power series they form an elegant hypergeometric function.

It is comforting to see how mathematics has been able to absorb the often dizzyingly imprecise heuristics of quantum physics and string theory, and to transform these intuitions into rigorous statements and proofs. Remarkably, these proofs have often *not* followed the path that physical arguments had suggested. It is not the role of mathematicians to clean the dishes of the physicists! On the contrary, in many cases completely new lines of thought had to be discovered in order to find the proofs, as was the case for mirror symmetry.

Niels Bohr was very fond of the notion of complimentarity. Originally this related to the fact that in quantum mechanics an electron could be viewed either as a particle or as a wave, but not both at the same time. The 'correct' point of view — particle or wave — is solely determined by the nature of the question, not by the nature of the electron. In his later years Bohr tried to push this idea to a much more embracing philosophy; one of his favourite complementary pairs was truth and clarity. Perhaps the pair of mathematical rigour and physical intuition should be added as another example of two mutually exclusive qualities. From that perspective, Plato's cave should be updated. Traditionally, this is the place where precise mathematical objects project vague shadows in the physical world. However, in the 'quantum cave' physical theories, which are often far from completely understood, project razor-sharp mathematical shadows and precisely formulated conjectures, which can be checked and proved.

From a wider perspective, beyond the traditional view of theoretical physics, it is clear that mathematics has a unique role to play in the rise of interdisciplinary research. It is ideally positioned to bring disjoint fields of research into contact. There is a charming technical term used in the study of lattices and sphere packings that indicates the number of spheres a given sphere touches: the kissing number (the root lattice of the Lie algebra E8 and the Leech lattice have for example exceptionally high kissing numbers). Perhaps mathematics as a whole can be characterized as a science with an exceptionally high kissing number.

This phenomenon is not only at work within the sciences as a whole but also in mathematics itself, although this is not universally acknowledged and taught to our students. In his powerful essay 'On proof and progress in mathematics' (*Bull. AMS* **30**, 2 (1994) 161–177). Thurston raises the important question of the 'right' definition of a mathematical concept. He illustrates this with the idea of a derivative. Of course there is the familiar $\varepsilon\delta$-definition that scares every first year analysis student. But Thurston raises the issue that there are many other points of view that are equally worthwhile. For example the dimension of time. Everyone has a natural idea about rate and velocity, even near instantaneous — children have no trouble saying 'for a moment I went very fast'. In fact, one can even 'feel' higher derivatives. The second derivative of constant acceleration pushes us deeper into our car seats; the third derivative or 'jerk' induces motion sickness. Equally important of course is the geometric way of thinking, which immediately suggests generalizations to more variables or higher dimensions. Thurston lists many more aspects: linear, infinitesimal, symbolic, approximate. In the end we need all these definitions to bring to life the full concept of a derivative. Deep mathematical notions are therefore like many faceted diamonds. The connection to quantum physics is just another valuable addition.

In this way the essence of mathematics is captured very well by the humble symbol of the equals sign in our equations. This interconnectedness of mathematics, both internally and externally, is perhaps difficult to explain to a general audience, but it is easy to hear! Just pronounce an equation. You will notice that 'is' constitutes the only verb in the sentence. It is an activity, as one has to actively connect A and B. That is an interesting comment on the purpose of this congress and on the nature of mathematics itself.

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Gene Golub Around the World Day

Gene Golub (1932–2007) is one of the founding fathers of numerical analysis. His book Matrix Computations, coauthored with Charles F. Van Loan, can be found in every mathematics library, and his algorithms, in particular the algorithm he designed together with William Kahan in 1970 for computing singular values, have opened many new areas of scientific computing. He had an enormous number of coauthors, and has been known personally by a lot of mathematicians all around the world. To reminisce his warm personality, one has chosen to celebrate his birthday all over the world.

Friday 29 February 2008 was a very special day. On this day at least 1,000 people at 32 events around the world celebrated the 19th birthday of Gene Golub for 48 hours. Gene Golub, a well-known numerical linear algebra scientist, passed away on 16 November 2007 at the age of 75 years. It seems that the text above contains a number of contradictions, however careful reading shows that there are no errors. The Gene Golub DCSE (Delft Center for Computational Science and Engineering) Symposium [1] was part of the Gene Golub Around the World Day. This event was one of a series of similar events replicated around the world on the same day in Adelaide, Berlin, Delft, Hong Kong, Leuven, Mexico, Moscow, Oxford, Rio de Janeiro, Shanghai, Stanford, Tsukuba, Urbana-Champaign, etc.

Figure 1 Gene Golub

Personal reminiscences

In the DCSE symposium a number of international speakers presented their research and its relation to the work of Gene Golub. The subjects of the talks: singular value decomposition (SVD), Google's PageRank, preconditioned conjugate gradients, orthogonal polynomials and the interior-point method, gave a nice overview of the influence of Gene Golub on these different subjects. Most of the speakers gave personal memories of their collaborations with Gene. The main speaker, Professor Bernd Fischer (Universität zu Lübeck) explained that numerical linear algebra tools developed in collaboration with Gene are crucial in medical applications. He specifically talked about the removal of tumours from liver tissue. Before medical intervention a picture of the liver is made by a CT scan. However, during the intervention the liver is transformed. Fischer applied his mathematical tools to change the original picture to the transformed one in order to enable the medical doctor to remove the correct part of the human liver.

Achievements

Born in Chicago, Gene was educated at the University of Illinois at Urbana-Champaign, receiving his BS (1953), MA (1954) and PhD (1959) all in the field of mathematics. His MA degree was more specifically in mathematical statistics. His PhD dissertation was entitled *The Use of Chebyshev Matrix Polynomials in the Iterative Solution of Linear Equations Compared to the Method of Successive Overrelaxation* and his thesis advisor was Abraham Taub. He had been at Stanford University since 1962 and became a professor there in 1970. In total, he had about 30 PhD students and 80 academic grandchildren. Gene had been awarded several honorary degrees from a variety of institutions.

The book *Matrix Computations of Golub and Van Loan* is very influential in the domain of numerical linear algebra. The book now has three editions, has sold over 50,000 copies and has been cited over 18,000 times. Its fourth edition is still in preparation (by Charles Van Loan). Besides the book, Gene had around 20 other papers with 100 or more citations and 100 papers with 20 or more citations. He has more than 200 coauthors!

Gene had a tremendous record of service to the whole scientific computing community. He served as the President of SIAM and played an important role in the foundation of the International Council for Industrial and Applied Mathematics (ICIAM). At the ICIAM 2007 congress, he was one of the originators and the chair of the workshop 'Future Directions in Numerical Analysis'. He is the founding editor of two important SIAM journals: the SIAM Journal on Scientific Computing and the SIAM Journal on Matrix Analysis and Applications. Gene founded the NA-NET and the NA-Digest, which are important electronic networks within the numerical analysis community.

Until the end of his life Gene had an open mind and liked to collaborate with many people. Gene praised his students and said they had made him a better person. He admired their personalities, behaviour, scholarship and integrity, and said he felt fortunate that he had met many of them. The living memory of him passes but the algorithms he created will continue to be used, probably indefinitely.

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1 ta.twi.tudelft.nl/nw/users/tang/genegolub

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History The golden age of mathematics

Stevin, Huygens and the Dutch republic

Mathematics played its role in the rise of the Dutch Republic. Simon Stevin and Christiaan Huygens were key figures in this. Though from very different backgrounds, both gave shape to Dutch culture. Fokko Jan Dijksterhuis, who recently received an NWO-vidi grant for his research on the history of mathematics, discusses their influence on mathematics and its historical significance.

The 17th century was the golden age of the Dutch republic. A small state lacking substantial natural resources, from the 1580s onwards it quickly became a wealthy trade centre that played a leading part in international politics. It produced cultural tour de forces in the arts, sciences and technology. Mathematics is represented by Simon Stevin (1548–1620) and Christiaan Huygens (1629–1695), spanning the main part of the golden age.

Figure 1 Christiaan Huygens, pastel drawing Bernard Vaillant, 1686. Collectie Huygensmuseum Hofwijck, Voorburg

Recent developments in cultural history have opened new perspectives on the mathematical pursuits of the Dutch republic.[1] Understanding the brilliant ideas of Stevin and Huygens in their historical context promises a new and deeper insight into their achievements. Stevin and Huygens lived at different stages of the golden age and their mathematics likewise differed. Stevin witnessed the birth of the Dutch republic and played an active role in its formation. Half a century later, the wealth, the power and the splendour of the republic had been established (and had even begun to wane) with Huygens as its main mathematical representative. The hands-on mathematics of Stevin differed greatly from Huygens' aristocratic geometry and yet, as will be argued here, they had essential features in common. In this article, the work of Stevin and Huygens will be sketched as part of, and giving shape to, the golden age.

Figure 2 Simon Stevin, Universiteitsbibliotheek Leiden

Mixed mathematics

A history of mathematics in the golden age begins with the realisation that early modern mathematics was an endeavour quite different from its present form. Viewed with our modern conception of mathematics, our main heroes Stevin and Huygens instantly disappear from the history of mathematics. Their contributions to mathematics per se were minor (Huygens) or virtually zero (Stevin). More importantly, by trying to single out their mathematical achievements we lose sight of the structure of their work and consequently their ingenuity. The first thing we need to realise is that the idea of pure mathematics dates from the early nineteenth century. Prior to Lagrange's and Cauchy's rationalist purification of mathematics, the whole range of mathematical sciences (geometry, arithmetic, astronomy, music, optics, etc.) were seen as *parts* of mathematics. The idea of an a priori given pure mathematics applied to physical domains simply did not exist. In early modern conceptions 'mathematica pura' was distinguished from 'mathematica mixta', indicating the level of abstraction of the object of study.[2] Geometry and arithmetic studied quantity as subsisting in itself; astronomy, music and optics studied quantity as subsisting in matter. The mixed parts were not separated, let alone subordinate to the pure parts of geometry and arithmetic: mathematics consisted of the study of reality in its quantitative aspect. From this historical point of view, Stevin and Huygens were full-blown mathematicians, and brilliant at that.

The beginning of the Dutch Golden age

When the northern Netherlands in 1579 revolted against Spanish rule and liberated themselves, a 50-year exodus of capital, intellect and skills from the southern Netherlands began. Although his motives are unclear, Stevin moved too and he turns up in Leiden in 1581, where he enrolled at the university in 1583. He had been born in Brugge in 1548, illegitimate son from a well-to-do family, and had worked as a bookkeeper in Antwerp and at the Brugge tax office.[3] Although little is known of his scholarly and practical education, the documented activities in his early years in Holland show that he was both literate and skilled, and that he applied himself to the whole spectrum of mathematics. During the 1580s, he published a range of books in Dutch, French and Latin, applied for patents for the construction of mills and performed experiments. These activities on the one hand reflected his previous experiences and on the other hand opened up new areas of interest, areas that fitted particularly well with his new environment. During these early Leiden years Stevin also developed relationships with notable Dutchmen, for example the future Delft burgomaster Johan Cornets de Groot (1554–1640) with whom he collaborated on mill projects, and the future stadholder Maurits van Nassau (1567–1625) who studied at Leiden university at the same time. In short, we see a Flemish exile in his thirties who energetically and quite successfully pursued his interests in matters mathematical.

Simon Stevin

The publications that reflected Stevin's previous occupations were *Nieuwe Inventie van rekeninghe van compaignie* (New invention of business calculation, 1581) and *Tafelen van Interest* (Tables of Interest, 1582).[4] *De Thiende* (The Dime, 1585) also built on his bookkeeping experience but had a much wider scope. It was an inspired plea for the use of decimal fractions in all calculating professions: "stargazers, surveyors, measurers of carpet, of wine, of bodies in general, cashiers, and all merchants". With works like these Stevin contributed to the growing amount of calculating in all kinds of professions. He did so by proposing specific methods and elucidating their character and application — to the point of disclosing professional secrets. Twenty years later Stevin got the opportunity to put his ideas about sophisticated bookkeeping into practice when he became administrator of Maurits' domains. He was given permission to introduce Italian double entry bookkeeping. In *Vorstelicke bouckhouding* (Royal bookkeeping, 1608) he described and explained his ideas and methods.

Stevin has achieved lasting fame with *De Beghinselen der Weeghconst* (Elements of the Art of Weighing, 1586), appended with *De Weeghdaet* (The Act of Weighing) and *De Beghinselen des Waterwichts* (The Elements of Waterweight). These publications reflect the interests of his new country, of machines and of water management. Stevin employed his findings in new mill designs, for which he got patents and which he realised in a couple of places — albeit not fully satisfactorily.[5] At the same time, however, these works reflected an international — most notably Italian — interest in the mathematics of mechanisms and the Renaissance of Greek mathematics. In particular the rediscovery of Archimedes had sparked a lively interest in this particularly 'tactile' mathematics and methods of analysis.

Stevin contributed to this Archimedean renaissance with an inventive inquiry into statics and hydrostatics that extended well beyond the original of Archimedes. The 'clootcrans' (wreath of spheres) is the most famous example. He proved by reductio ad absurdum that two bodies on an inclined plane are in balance when their weights are proportional to the length of the planes. Imagine a wreath of equal spheres at equal distances that can move freely. The number of spheres is proportional to the length of the sides of the triangle. If the spheres on the one side would *not* keep those at the other side in balance, the wreath would begin to move. However, it would soon reach the same state and thus a perpetual motion would occur. Because a perpetuum mobile is impossible, the supposition is false, and thus the spheres must be in balance. Stevin was so proud of this proof that he used it as the frontispiece of the *Weeghconst*.

With the *Weeghconst* Stevin expressly presented himself as a man of learning, displaying his knowledge of mathematical literature and elaborating his account in geometric fashion by formulating principles, drawing up thought-experiments and deducing propositions. He did so, however, in an idiosyncratic manner. He explicitly combined his Archimedean analyses with practical pursuits. The titles of his books reflect his views: the *elements* of weighing was appended with the *act* of weighing, and the elements of waterweighing contained a commencement of the act of waterweighing. He called these two aspects 'spiegheling en daet', reflection and action.[6] Both were requisite: just like a foundation without a building is futile "so the reflection on the principles of the arts is lost labour where the end does not tend to the action".[7] In all his work Stevin consequently combined practical work and literary pursuits. In bookkeeping or mill building he did not content himself with getting the paperwork right and putting machines to work. He also committed his opinions and inquiries to paper, learnedly elaborating them.

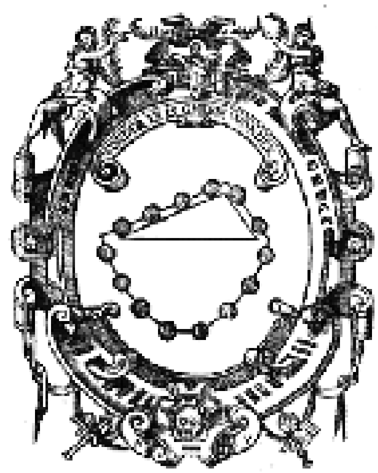


Figure 3 The 'clootcrans'

'Spiegheling' and 'daet' are usually interpreted as 'theory' and 'practice' but we should be careful interpreting these words with our modern understanding. By 'spiegheling' Stevin understood the intellectual analysis aimed at disclosing principles and causes that should precede the 'daet' of understanding and employing phenomena in the real world. The *Weeghconst* and the *Weeghdaet* both contain theories of weighing, with principles, deductions and so on. The former considers weights per se whereas the latter considers them in their physical appearance, Stevin said.[8] For us (early 21st-century mathematicians) such a distinction is not self-evident but Stevin's explanation corresponds quite well with the distinction 'pure' and 'mixed'. This is remarkable, because 'spiegheling en daet' may best be translated

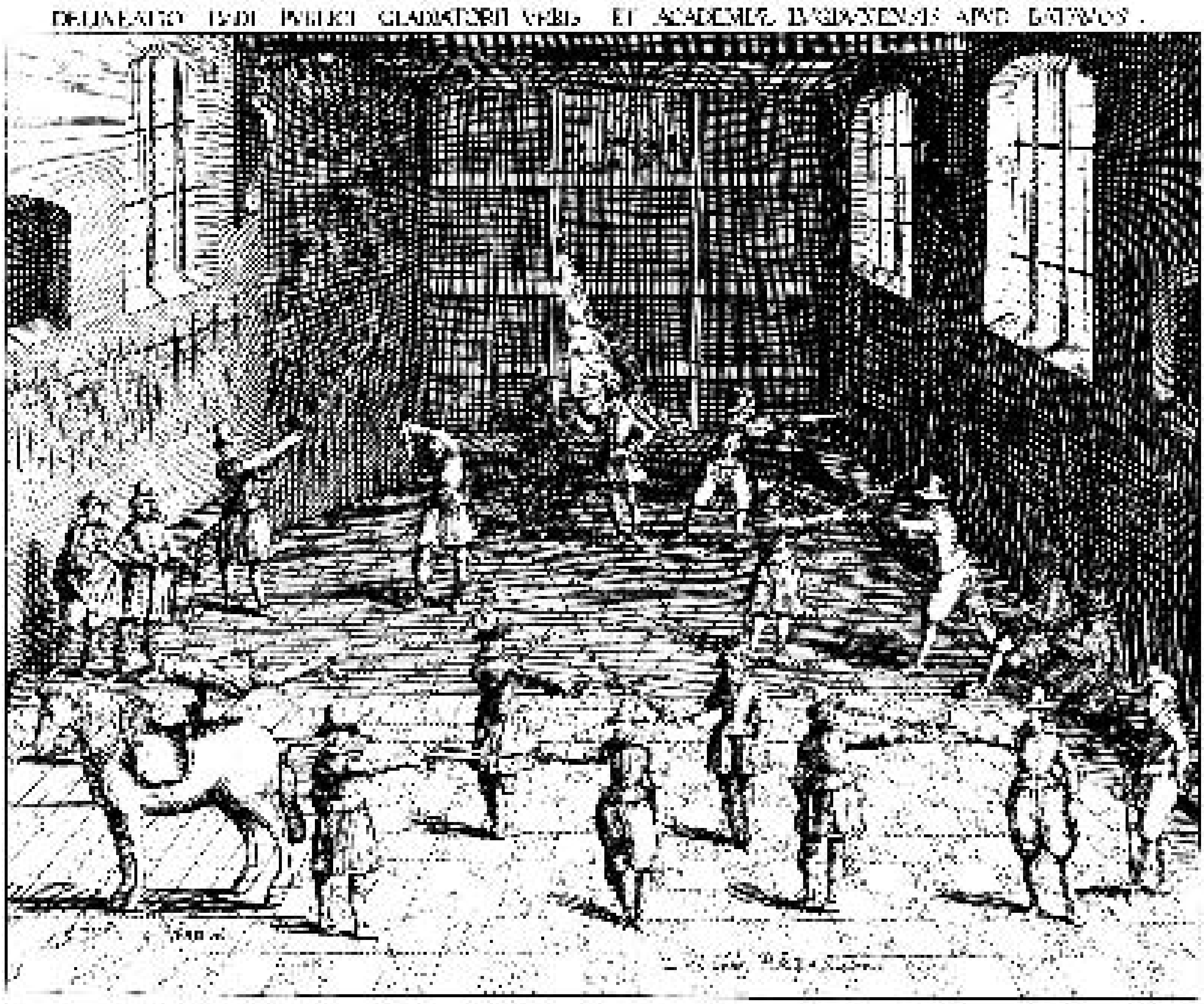


Figure 4 The 'Duytsche Mathematique' found its first home in the fencing school of its first professor, Ludolf van Ceulen, on the ground floor of the University Library. This picture nicely illustrates the close links that existed between mathematics and the military at that time. Copper engraving: Jan Cornelisz. van't Woudt (1610)

as 'reflection' and 'action'. That was a second distinction drawn in early modern mathematics, signifying the goal of study rather than the object.[9] Stevin's oblique use of reflection and action may be explained by realising that he was primarily interested in the mathematics of an artisan — bookkeeping, machines and, later on, fortifications and navigation — to which he applied his considerable speculative ingenuity in finding principles and causes. The mathematics of natural phenomena considered outside a context of artefacts and 'practices' lay beyond his interest.

The use of the vernacular was important for Stevin. Not only did he consider Dutch the best way to reach his intended audience (men of action), he even went so far as to regard Dutch the language best suited for learning. The *Weeghconst* opened with a long digression on the virtues of Dutch, being more clear and more efficient than any other language. He rejected the tendency of other writers in the vernacular to Dutchify Latin words and proposed a whole list of truly Dutch words. Irrespective of the virtues of Stevin's linguistic theories, he did give Dutch its own word for mathematics: wisconst, nowadays wiskunde, literally 'art of knowing'.

The establishing of the Leiden engineering school in 1600

Stevin's greatest success in realising his ideas was the establishment in 1600 of the 'Duytsche Mathematique', the engineering school at Leiden University. It would train fortificationists with precisely the combination of 'spiegheling en daet' Stevin envisioned in his writings. The teaching was in the vernacular, hence *Duytsche* mathematics. Although Stevin never taught at the engineering school, he drew up a detailed curriculum in which a careful selection of mathematical theory was combined with field practice. For example:

"The measuring of circles with segments of that sort, further the area of spheres. The shapes named ellipsis, parabola, hyperbola and the like, that is not necessary here, because engineers are very seldom made to perform such measurements; but only they shall learn with straight planes, after that curvilinear in surveyor's manner, measuring thus a plane by various division, like in triangles or other planes to see how this matches with that." [10]

as a victory of Stevin over the practically inclined engineers in the Dutch army. The idea that engineers should also learn from books was not obvious, and would not be for at least a century.

Stevin's success can be explained by his close ties to Count Maurits, the Dutch stadholder and army commander.[11] They probably knew each other from their student year in Leiden in the 1580s, and in the 1590s they established a close relationship. In 1593 Stevin entered the service of Maurits as quartermaster of the army.[12] From this time his publications dealt with issues of state building and reinforcement, for example *Sterctenbouwing* (Fortification, 1594) and *De Havenvinding* (The finding of harbours, 1599). These had been preceded by a treatise on citizenship *Vita Politica. Het Burgerlick Leven* (Political Life. The Civil Life, 1590). Stevin's ideas and pursuits fitted particularly well with his employer's interests. Maurits was greatly interested in mathematics and shared Stevin's idea of combining practical improvements with theoretical reflection. The two had lengthy conversations on these matters, which Stevin eventually put down in his *Wisconstighe Ghedachtenissen* (Mathematical Memoirs, 1605–1608). These reveal several original contributions of Maurits.[13] Maurits played a central role in the military reforms that turned the Dutch army into a highly skilled, professional and quite successful force. Part of these reforms was the 'Duytsche Mathematique' that Maurits installed highhandedly at Leiden university.

The promotion of mathematics in warfare, state building and land development was not just a Dutch affair. The second stadholder of the republic Willem Lodewijk of Friesland had similar ideas about the intellectual and material reinforcement of the new state. His scholarly interests were mainly aimed at classical military theories but he provided for the cultivation of mathematics by appointing Adriaan Metius in the chair of mathematics at Franeker University. Like Stevin, Metius disclosed new developments in the mathematical sciences to a broad audience by publishing a range of books in both Dutch and Latin.[14] In Franeker as well as Leiden, mathematics figured next to (Calvinist) theology as the means of intellectually reinforcing the new republic. Stevin's religious inclinations are somewhat diffuse but we have acquired a good understanding of his mathematical persona. His specific and explicit mixing of reflection and action resulted in a series of textual and material works in which he ingeniously extended the renaissance mathematics inspired by Archimedes. The beautiful Dutch word 'vernufteling' is most appropriate for Stevin, indicating cleverness of both hand and mind.

French mathematics brought to Leiden

The historical link between Stevin and Huygens is established by Frans van Schooten Jr. (1615–1660). Van Schooten's career reflects the transition from the Dutch-spoken, utilitarian mathematics of Stevin to the Latin-written, academic mathematics of Huygens. After his studies at Leiden university, he began replacing his father, Frans van Schooten Sr., the then professor of 'Duytsche Mathematique' in the 1630s, before succeeding him in 1645. In the meantime, he had established himself as an advocate of the new mathematics coming from France. He gathered and published papers by Fermat and Viète and collaborated with René Descartes. He commented on the manuscript of *La Géométrie* and made the illustrations. His claim to fame is *Geometria a Renato Des Cartes*, a Latin translation with commentaries and supplements disclosing Descartes' new geometry to the international mathematical community. A first edition appeared in 1649, followed by an expanded edition in 1659–1661. This edition contained contributions from elite students Van Schooten had attracted to Leiden: patrician sons like Johan de Witt (future governor of the republic), Johannes Hudde (future burgomaster of Amsterdam) and Christiaan Huygens.

Christiaan Huygens

It remains one of the miracles of the golden age that patricians like De Witt and Hudde developed and cultivated an interest in mathematics, a field of study that apparently bore no relevance to their administrative ambitions. Their amateur interest in the mathematical sciences may be a major key to understanding the unprecedented rise of the epistemic and cultural status of mathematics in the 17th century. In the course of the scientific revolution, mathematics rose from being an art pursued by ingenious professionals to the language of philosophy — and even nature — cultivated by savants of all ranks. The growing interest in natural inquiry among Dutch patricians has been explained by the 'aristocratisation' of this group. They were no nobles but they increasingly took up noble activities like investing in landed property, building dynasties of power, and taking up arts and letters.

Huygens too was an amateur in mathematics; until the 19th century any true scientist was. His case is different, however, from his fellow students. He managed to devote his entire life to his main pleasure: mathematics. Huygens was the second son of Constantijn Huygens, secretary of the stadholder and a leading man of learning and art in the Dutch republic.[15] His predilection for mathematics showed at an early age, when he and his older brother Constantijn were educated by private tutors. He helped his brother with mathematical exercises and was soon bored with writing poems. After the brothers had gone through university, their father could set out to plot a diplomatic career for his sons. However, in 1650 the first stadholderless period began and with the Oranges out of power no posts were available to the Huygenses. Huygens was free to pursue his interests (and happy too, we may presume). He would do so for the rest of his life, even when his brother made a diplomatic career some years later. For a man of learning of Huygens' rank few public positions were available and in the republic next to none. A university professorship was far below his standing and the republic did not have a lustrous court. The only options were the great courts of Italy, France and England. His bid for Florence was unsuccessful but in 1666 Colbert brought him to Paris to chair Louis XIV's Académie des Sciences. He would stay there until the early 1680s, spending his last years at his family residences in The Hague and Voorburg: a life in mathematics, free from public obligations — a savant's paradise.

What did this pastime look like? In the course of his life, Huygens worked on a wide range of topics: the catenary and floating bodies; Cartesian ovals, quadrature and rectification; lenses, telescopes, heavenly observations and light waves; impact, fall, circular and pendulum motion, clocks and longitude; probability; tuning; and so on. Historians have lamented that Huygens' oeuvre lacks a unifying idea, that it is an eclectic collection of puzzles that came his way through coincidental encounters and the like. From the viewpoint of early modern natural philosophy and in comparison with men like Galileo, Descartes and Newton, Huygens' oeuvre indeed looks fragmentary. Yet, Huygens had no taste for philosophy, as his pupil Leibniz would aptly observe. He was a mathematician. From the perspective of early modern mathematics, Huygens' oeuvre does not look incoherent at all. All his activities were mathematical and he covered almost the entire range of the mathematical sciences: geometry, (hydro)statics, optics, astronomy, mechanics, arithmetics and music. Furthermore, and this is important to realise, he barely went beyond the domain of mathematics, except for one important case that will be discussed below. In other words Huygens was a full-blown, 17th-century mathematician.

Huygens' first steps in the Republic of Letters earned him the epithet 'little Archimedes'. A discussion of projectile motion greatly impressed Mersenne, who wrote to Huygens' father saying that his boy would soon become a new Archimedes.[16] From then on, Constantijn would call Christiaan his little Archimedes.

Huygens' style was indeed Archimedean. He had a visual and tangible approach, in which he *saw* the ratios fundamental to classical geometry and favoured the kinematic understanding of curves Van Schooten had taught him in his introduction to Descartes' new geometry. His theory of evolutes is based on the actual motion of a thread unwinding from a curve. Archimedes was also prominent in Huygens' first publication in 1651, a treatise on quadrature in which he revealed mistakes in a recent treatise on the quadrature of the circle. He proposed an improved method and applied this to conic sections as well. He showed how to limit the amount of calculations for the method of exhaustion by using the centre of gravity of line segments. The visual and tangible style of Huygens' earliest work would be a distinctive feature of his mathematics.

Lenses and telescopes

In 1653 Huygens became 'totally absorbed' by dioptrics, the mathematical analysis of lenses and their configurations.[17] Although the sine law had been published by Descartes in 1637, Huygens was the first to apply it to lenses actually used in telescopes. Descartes had only been interested in determining the shape of lenses with a perfect focus, finding them elliptic or hyperbolic. Those lenses, however, turned out to be very difficult to manufacture and telescopes were fitted with ordinary spherical lenses. In a study that would be published only posthumously, Huygens employed the sine law to derive the properties of lenses in a rigorous fashion. Due to spherical aberration the focus is not perfect and Huygens defined it as the limit of intersections of the axis. He then derived exact expressions for the focal distance of any kind of lens: plano-convex, plano-concave, bi-convex, etc. Only then did he show how these cumbersome expressions reduced to the simple lens formulas when the thickness of the lens is neglected.

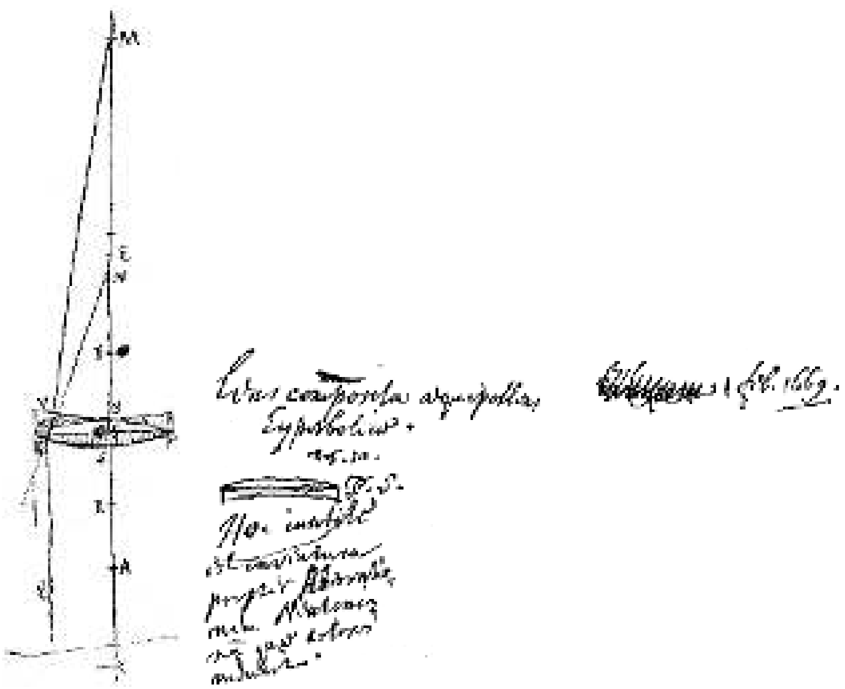
Huygens' theory of the telescope reveals a second feature of his mathematics: the rigorous elaboration of mathematical problem focused on concrete topics. His oeuvre consists of a couple of clusters of topics that are bound together by a marked interest in instruments. This is not to say that material problems guided his mathematical studies. He valued the mathematics of things greatly, rigorously elaborating the dioptrics of lenses. Still, he selected mathematical topics for their relevance to understanding the telescope, leaving out sophisticated problems like obliquely incident rays.

After he finished his theory of the telescope, Huygens turned to the practice of lenses. He inquired about the grinding and polishing of lenses and began manufacturing telescopes with his brother Constantijn. In 1655 he discovered a satellite of Saturn with one of his telescopes, the first new body in the solar system since Galileo's famous discoveries in 1610. Soon after he was able to figure out the cause of the strange shape of Saturn: the planet was surrounded by a flat, circular disc. These discoveries gained him fame, but there is something odd about Huygens' use of the telescope. Apart from these and a couple of other observations, Huygens never became a systematic astronomical observer. His interest in the telescope on the other hand was virtually unlimited. He endlessly manufactured lenses and telescopes, developing configurations with excellent optical properties and inventing improvements like the diaphragm and a rudimentary micrometer. With instruments so central to Huygens' mathematics it is remarkable how little interested he was in their actual employment, as though the tinkering with them, both intellectually and materially, sufficed.

In the 1660s he continued his mathematical analysis of lenses by elaborating on a theory of spherical aberration, relating the amount of aberration to the properties of a lens. He then managed to determine a configuration of two lenses that mutually cancelled out their spherical aberration. 'Eureka', Huygens exclaimed on 1 February 1669 in Archimedean style — a perfect telescope made of ordinary, spherical lenses. Soon, the novice Newton would prove that his invention would never work in practice due to the nature of chromatic aberration. On 25 October 1672, Huygens crossed out his invention and wrote 'useless'. This debacle is probably the reason he never published his dioptrics: a theory without an impressive invention was futile.[18]

Mechanics

In mechanics things were different. Huygens gained fame with the invention in 1657 of the pendulum clock and this was a perfect stepping-stone for publishing his mathematics. That winter he invented a mechanism to regulate a clock with a pendulum. Mechanical clocks were grossly inaccurate at that time.



Left, Huygens's design of the lens (1669); right: the word Eureka is crossed out.

The motion of a pendulum was known to be very regular and used by astronomers and the like to measure time. The combination of these two elements yielded a clock that was accurate within a couple of seconds a day. In 1658 Huygens published *Horologium*, a description of his clock, that contained an additional invention. He hung the pendulum between little 'cheeks', so that its length diminished for larger displacements. A pendulum was, after all, not isochronous: the period increases slightly with the amplitude. Modifying the path of them both slightly would yield an isochronous pendulum and thus a perfect clock. However, Huygens did not know the exact path required (or the shape of the cheeks) and for this he needed a thorough understanding of pendulum motion.

Around the same time Huygens had begun to study free fall and in 1659, his annus mirabilis, he tackled a whole series of problems relating to accelerated motion.[19] The problem was to determine g, the constant of gravitational acceleration, or in 17th-century phrasing: the distance an object in free fall covers in one second. Several experiments had been performed without a conclusive outcome. Huygens himself tried to measure the distance by letting a seconds pendulum hit the wall and having a mass hit the ground at the same time. It did not yield any convincing results so he turned to a mathematical analysis of the situation. A pendulum is a body in circular motion and weight, he argued, is identical to the tension exerted by the body on the cord. Comparing a horizontally moving body deflected by gravity (a parabolic path) and by a chord (a circular path) and noting the identical centres of curvature, Huygens derived a measure for centrifugal acceleration equal to our modern mv^2/r. He then devised a set-up in which gravitational and centrifugal acceleration are in balance: a pendulum rotating in a horizontal plane. Knowing now how to calculate the centrifugal acceleration of this conical pendulum, Huygens could determine gravitational acceleration to be 9.79m/s^2 (in modern terms) — quite accurate indeed!

Huygens then attacked the problem of the isochronous pendulum and this shows all the features of his mathematical genius.[20] What is the ratio between an oscillation with a small amplitude EZ and free fall over the height of the pendulum TZ? Huygens compared the infinitesimals at E and B and constructed the parabola $AD\Sigma$ representing velocities along the path of the pendulum. He then considered the times to traverse the

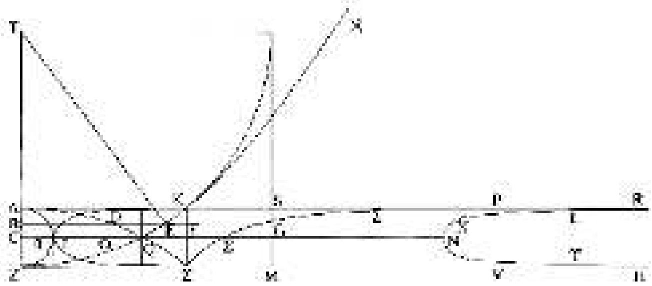


Figure 5 Top: the deduction of the isochronism of the cycloid (*Codex Hugeniorum* **26**, f.72r.). Bottom: diagram of this figure (construction taken from: Joella G. Yoder, *Unrolling Time. Christiaan Huygens and the Mathematization of Nature* (1989), Cambridge, Cambridge University Press)

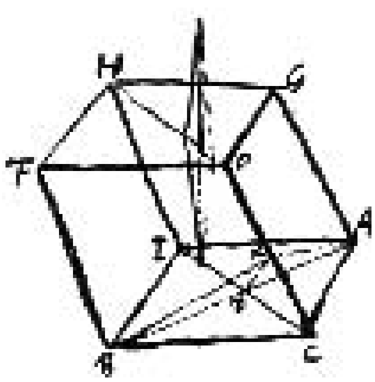
infinitesimals and constructed curve LXN. Notice that the curves are constructed within the diagram and directly represent the mathematics of the motions. The time to traverse arc KEZ is equal to the indented rectangle between AZ and LXN, but this involved the uniform velocity over AZ. To get around that, Huygens approximated arc KEZ with a parabola $ZK\aleph$ and determined the time anew. He concluded that the time is independent of the amplitude, and so he established isochronism. However, this applied only to small amplitudes. Huygens then showed his mathematical ingenuity and his masterly eye in geometry. He went through his diagram again and found the solution by recognizing the properties of the curve needed. The radius TE is perpendicular to the circle ZEK but not to the parabola $ZK\aleph$ he had used as an approximation. He needed a new curve that did preserve the conditions of his derivation. And then, out of the blue, Huygens said: "I saw this was the cycloid because of the familiar method to draw its tangent". Thus he found out that an isochronous pendulum should follow a cycloidal path. And it turns out that the involute of a cycloid is also a cycloid. So the cheeks of his clocks should be cycloids. In 1673 Huygens published these results in his magnum opus *Horologium Oscillatorium*, in which the mathematics of motion was presented with the construction of his ingenious invention.

The Huygens principle

Huygens' work on the kinematics of fall, circular and pendulum motion has all the characteristics of his mathematics: exact, rigorous, visual and concrete. It was a brilliant specimen of the new Galilean science of motion. It was not revolutionary, though. Huygens did not break new grounds but stayed well within the established domains of 17th century mathematics. He consciously did so. In mechanics he explicitly rejected non-kinematic concepts like force. And in optics too, he kept to the beaten track when he said that the coloured fringes that disturbed his telescopic images exceeded mathematical analysis.[21] Traditionally colours were not subject to mathematical analysis and Huygens was not someone to change that. Newton would turn the science of colours 'mathematical' with his prism experiments. And he would turn mechanics into dynamics by introducing a new concept of force. Still, Huygens would go on to make a revolutionary step beyond the established borders of mathematical inquiry; however, he did not truly realise he was doing so. In the 1670s, Huygens tackled an intriguing optical phenomenon: the strange refraction of Iceland crystal.[22] This crystal displays a double refraction and one of these does not follow the sine law of refraction: a perpendicular ray is broken and an oblique ray passes unrefracted. Huygens was by then convinced that light consists of waves in a material ether but he could not figure out how a wave traversing a refracting surface in a straight direction was diverted to produce a refracted ray. The flash of genius came in August 1677 and the way it came corroborates the thorough mathematical character of Huygens' work. He was working on a sophisticated dioptrics topic that also seemed to defy his understanding of waves: caustics, the bright curves produced when light passes through lenses and other curved bodies (like a glass of water in the sun). Trying to construct geometrically a wave propagating after traversing the curved boundary, Huygens hit upon an ingenious idea that we now know as his principle of wave propagation. At this time it was only a tiny sketch jotted down to note the method he used in constructing caustics by means of tangents to wavelets. And then to tackle strange refraction, without pausing to elaborate on his idea, Huygens went on to construct a wave refracted in Iceland crystal. He had it propagate unequally in the crystal producing an elliptical instead of a circular wave and this did the trick. The tangent to these ellipses was parallel to the refracting surface but propagated somewhat askew and thus the ray was refracted. 'Eureka', Huygens noted again and this time nobody would take it away from him.

Unwittingly, Huygens took a revolutionary step when he tackled strange refraction. His principle of wave propagation made mathematical the properties of hypothetical entities in the unobservable realm of the ether; he had transgressed the borders of mathematics, for mathematics did not treat the unobservable motions of unobservable matter that explained natural phenomena. Huygens thus transferred mathematical inquiry to the domain of natural philosophy and he was the first to do so. Yet to Huygens this went without saying: of course the motions of hypothetical waves in the ether ought to be discussed in the same exact manner as the motions of rays, pendula and billiard balls. Bear in mind that the principle came from mathematics rather than physics. Huygens devised a way to geometrically construct caustics and strangely refracted rays; he did not consider the nature of light waves at this point. He would do so only afterwards when preparing a presentation of his account for the Académie, which he eventually published in 1690 as *Traité de la Lumière*. It presents the principle of wave propagation as a much better means of deriving the laws of optics and as a unique way of enabling the penetration of strange refractions. But it does not present it as we see it: a revolutionary transformation of mathematics (and natural philosophy). If Huygens was a revolutionary, he was sleepwalking.

Huygens' wave theory was part of his dioptrics. Although he never published his theory of the telescope, he cut the umbili-



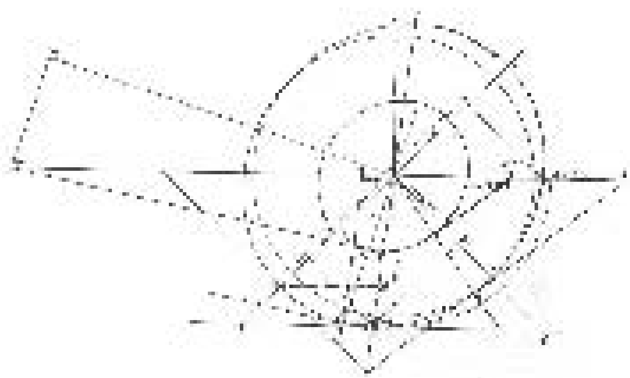


Figure 6 Strange refraction. On the left, Huygens's sketch of this phenomenon (1672); middle: his sketch of the 'Huygens principle' (1677); right: the solution to the problem

cal cord only at the very last. Until 1690, he planned a book on optics that included both, eventually deciding to publish his wave theory separately [23]. Thus this cluster of activities was pulled apart, obscuring the coherence of telescopes and waves. Light waves were part of Huygens' mathematical consideration of telescopes that had first produced his dioptrical analysis of lenses.

In Huygens we see a similar combination of 'spiegheling en daet' as in Stevin. Yet the nature and the context of his mathematics was entirely different. Huygens was a 'vernufteling' like Stevin, but in the learned world of instruments, academies and letters, rather than the artisan's world of mills, bookkeeping and fortifications. Huygens tinkered with instruments for the sake of tinkering with them, as an amateur pastime. He could be called a mathematical virtuoso but Huygens may have been somewhat ill at ease with such a qualification. He saw himself as a physicomathematician like Galileo; consider the evaluation of 17th century savants he wrote at the end of his life [24]. His father would surely have balked at the idea. He once received a letter that praised his 'mathématicien' son. Constantijn replied irritably that he did not know that he had craftsmen in his family. A *learned* mathematician was a 'géomètre' after all. For him, Christiaan always remained his little Archimedes.

Conclusion

Stevin and Huygens were children of their times and so were their mathematics. The times were changing and so were their mathematics. Early modern mathematics was a much different affair to mathematics today. Considering the pursuits of Stevin and Huygens from this historical perspective gives both an insight into their achievements and reinforces their ingenuity. Such are the fruits of recent developments in the historiography of mathematics. Inquiry into the culture of early modern mathematics raises exciting questions, for example on the role mathematicians played in the formation of the Dutch republic and the way patrician amateurs paved the way for the mathematization of natural philosophy. It promises to yield a better understanding of the way mathematicians began shaping our modern world.

Referenties

1 Apart from my own work, some examples are: Peter Dear, *Discipline and Experience. The Mathematical Way in the Scientific Revolution* (Chicago, 1995); Andrew Warwick, *Masters of Theory. Cambridge and the Rise of Mathematical Physics* (Chicago, 2003); Eric Ash, *Power, Knowledge, and Expertise in Elizabethan England* (Baltimore, 2004); Wolfgang Lefèvre (ed.), *Picturing Machines. 1400-1700* (Boston, 2004); Volker Remmert, *Widmung, Welterklärung und Wissenschaftslegitimierung: Titelbilder und ihre Funktionen in der Wissenschaftlichen Revolution* (Wiesbaden, 2005); Matthew Jones, *The Good Life in the Scientific Revolution. Descartes, Pascal, Leibniz, and the Cultivation of Virtue* (Chicago, 2006).

2 "With regard to their Object, *Mathematics* are divided into *pure* or *abstract*; and *mix'd*. Pure *Mathematics* consider Quantity, abstractedly; and without any relation to Matter: Mix'd *Mathematics* consider Quantity as subsisting in material Beings, and as continually interwove." Chambers, Ephraim, *Cyclopædia, or, 'An universal dictionary of arts and sciences : containing the definitions of the terms, and accounts of the things signify'd thereby, in the several arts, both liberal and mechanical, and the several sciences, human and divine : the figures, kinds, properties, productions, preparations, and uses, of things natural and artificial : the rise, progress, and state of things ecclesiastical, civil, military, and commercial : with the several systems, sects, opinions, &c : among philosophers, divines, mathematicians, physicians, antiquaries, critics, &c : the whole intended as a course of ancient and modern learning'*. The Second Volume (1728), 509.

3 Berghe, Guido Vanden, 'Simon Stevin. Een leven in de schaduw van de macht', pp. 19–25 in H. Elkadem (red.), *Simon Stevin 1548-1620. De geboorte van de nieuwe wetenschap*.

4 *Tafelen* was published by Plantijn in Antwerp (dedicated to the aldermen of Leiden), *Nieuwe Inventie* by Hendricsz in Delft (dedicated to the Amsterdam burgomasters). *Nieuwe Inventie* has only been recently discovered and now turns out to be Stevin's first publication. Heirwegh, Jean-Jacques, and Frédéric Métin, 'Simon Stevin en de financiële wereld', pp. 73–81 in *Simon Stevin*.

5 Dijksterhuis, E. J., *Simon Stevin. Science in The Netherlands around 1600*, Den Haag, 1970, pp. 93–98.

6 In the *Wisconstighe Ghedachtenissen* Stevin later elaborated his ideas about the 'mixing of reflection and action' (part of the introduction of the *Eertclootschrift* in the *Weereltschrift*, part 1 of the *Memoirs*). "Spiegheling is een verdochten handel sonder natuerlicke stof, ghelijck onder anderen sijn de Spieghelinghen des Spieghelaers *Euclides*, handelende deur stelling (per hypotesim) van grootheden en ghetalen, maer elck ghescheyden van natuerlicke stof. Daet is een handel die wesentlick met natuerlicke stof gheschiet, als lant en wallen meten, de menichte der roen of voeten tellen dieder in sijn, en dierghelijcke. T' besluyt vande voorstellen der Spieghelingh is volcommen, maer der daet onvolcommen: ..."

7 Weeghdaet, 3. "Alsoo is de spiegheling (theoria) inde beghinselen der consten verloren arbeydt, daer t'einde totte deat (effectum) niet en strect."

8 Weeghconst, 1.

9 "Mathematics are distinguish'd with regard to their End, into Speculative, which rest in the bare Contemplation of the Properties of Things; and Practical, which apply the Knowledge of those Properties to some Uses in Life." Chambers, *Cyclopædia*, 509.

10 Molhuysen, *Bronnen*, 390*. "Het meten des rondts mette gedeelten van dien aengaende, voerts het vlack des cloots, de formen genaemt ellipsis, parabola, hyperbole ende diergelijcke, dat en is hyer nyet nodich, wantet den ingenieurs seer selden te voeren compt, sulcke metinge te moeten doen; maer alleenlyck sullense leeren met rechtlinige platten, daer na cromlinige landtmetersche wijse, metende alsoe een plat deur versceyde verdeelinge, als in dryehoucken of ander platten om te syen hoe t'een besluyt met het ander overcompt."

11 After his father Willem van Oranje had been assassinated in 1584, Maurits became Stadholder of Holland and Zeeland in 1585, at his 18th birthday. Although he used the title 'Prince of Orange' from 1584, he officially became prince only in 1618.

12 Berghe, *Simon Stevin*, 23.

13 Heuvel, Charles van den, 'Wisconstighe Ghedachtenissen. Maurits over de kunsten en wetenschappen in het werk van Stevin.', pp. 107–121 in Kees Zandvliet, Maurits Prins van Oranje (Zwolle, 2000).

14 See: Dijksterhuis, F. J., 'Duytsche Mathematique and the building of a new society: pursuits of mathematics in the seventeenth-century Dutch republic' in L. Cormack (ed.), *Mathematical Practitioners and the Transformation of Natural Knowledge in early modern Europe (forthcoming)*.

15 A good, concise biography in Dutch is Vermij, R., *Christiaan Huygens. De Mathematisering van de Werkelijkheid*, Diemen, 2004. A highly readible, although somewhat fictitious biography in English is Andriesse, C. D., Huygens. *The Man behind the Principle*, Cambridge, 2005. Vermij does not go into the mathematics while Andriesse does in enlightening and inspiring ways.

16 Huygens, Chr., *Oeuvres Complètes de Christiaan Huygens*. 22 vols, The Hague, 1888-1950: vol. 1, 47.

17 Dijksterhuis, F. J., *Lenses and Waves. Christiaan Huygens and the Mathematical Science of Optics in the Seventeenth Century*, Dordrecht, 2004, pp. 11–24.

18 Dijksterhuis, *Lenses and Waves*, pp. 67–91.

19 Yoder, J. G., *Unrolling Time. Christiaan Huygens and the Mathematization of Nature*, Cambridge, 1989, pp. 9–32.

20 Yoder, *Unrolling Time*, pp. 44–63.

21 Huygens, *Oeuvres Complètes*, vol. 17, p. 359.

22 Dijksterhuis, *Lenses and Waves*, pp. 169–184.

23 Dijksterhuis, *Lenses and Waves*, pp. 219–223.

24 Huygens, *Oeuvres Complètes*, vol. 10, pp. 399–406.

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Society

Math Girls Rule
A Dutch success story

Mathematics according to the average Dutch: boring, uninteresting, unknown, unsexy, for nerds focusing on gaming, and most definitely not for girls. But out of the blue appeared an initiative that itself turned part of the tide: The Mathgirls blog.

During March 2006, the two Dutch female mathematics PhD students Jeanine Daems and Ionica Smeets started their 'mathgirls'-blog at www.wiskundemeisjes.nl. Who would have thought then that their original plan 'to just put up some of the titbits that we talk about over lunch in the math institute' would take off with such success? They tell stories of strange deaths of mathematicians, interview mathematicians about their favourite living colleagues, pick up movies and songs about mathematics from YouTube, tell jokes, give biographies and knitting instructions for Klein bottles, present puzzles and more. This irresistible potpourri of mathematics trivia — and not so trivia — has gained them national fame: they appear as celebs on TV shows and their blog recently received two prizes. With a small crew of other students, they write enthusiastically about mathematics in newspapers, on websites, etc. And as they are so knowledgeable and irresistibly gossipy, their information tends to be correct and to-the-point. Add to that their talent for writing. If you ask around, many people think that they have been there for much longer than just two years: they seem to have become part of the landscape in no time at all.

As a matter of fact, I am sure they are hanging around at 5ecm just now, so be aware: they might want to interview you for their next posting!

Their activity has significantly increased public awareness of mathematics (and even of the mere existence of mathematics, mathematicians and women in mathematics) throughout the country — as is shown by the comments to some of their posts.

The picture on the right shows the illustration on a T-shirt depicting the two ladies (schematically, that is) with their slogan 'Math Girls Rule' written across it. I have already successfully exported such a specimen to Japan. Many more colours and sizes are available on their web page, in addition to a neat poster with a similar design.

The blog itself is written in Dutch, so if this is not one of the languages that you have mastered, you'll have to create something similar yourself, I am afraid.

Weblog location: www.wiskundemeisjes.nl

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Research

Noncommutative geometry and number theory

Noncommutative geometry, the study of spaces with a not necessarily commutative algebra of coordinates, is a field that has emerged from theoretical physics. In recent years, it has also directed its efforts to arithmetical problems, including the study of the Riemann zeta function. In this article, Matilde Marcolli provides us with some impressions of this emerging field.

Noncommutative geometry is a modern field of mathematics begun by Alain Connes in the early 1980s. It provides powerful tools to treat spaces that are essentially of a quantum nature. Unlike the case of ordinary spaces, their algebra of coordinates is noncommutative, reflecting phenomena like the Heisenberg uncertainty principle in quantum mechanics.

What is especially interesting is the fact that such quantum spaces are abundant in mathematics. One obtains them easily when one considers equivalence relations that are so drastic that they tend to collapse most points together, yet one wishes to retain enough information in the process to be able to do interesting geometry on the resulting space.

In such cases, noncommutative geometry shows that there is a quantum cloud surrounding the classical space, which retains all the essential geometric information, even when the underlying classical space becomes extremely degenerate. It is to this quantum aura that all sophisticated tools of geometry and mathematical analysis, properly reinterpreted, can still be applied.

It has become increasingly evident in recent years that the tools of noncommutative geometry may find new and important applications in number theory, a very different branch of pure mathematics with an ancient and illustrious history. This has happened mostly through a new approach of Connes to the Riemann hypothesis (at present the most famous unsolved problem in mathematics).

Quantum computers

The first instance of such connections between noncommutative geometry and number theory emerged earlier, when Bost and Connes discovered a very interesting noncommutative space with remarkable arithmetic properties. The system it describes consists of quantized optical phases, discretized at different scales. These are essentially the phasors used in modelling quantum computers (see Figure 1). A mechanism that accounts for consistency over scale changes organizes the phasors via a kind of renormalization procedure. This consistency condition imposes the equivalence relation that makes the resulting space noncommutative.

The system obtained in this way has intrinsic dynamics, which makes it evolve in time, and one can consider corresponding thermodynamic equilibrium states at various temperatures. Above a certain critical temperature the distribution of phases is essentially chaotic and there is a unique equilibrium state. At the critical temperature the system undergoes a phase transition with spontaneous symmetry breaking and below critical temperature the system exhibits many different equilibrium states parameterized by arithmetic data.

Especially interesting is what happens at zero temperature. There the arithmetic structure that governs the action of the symmetry group of the system on the extremal ground states is the same one that answers the famous mathematical problem (solved by Gauss) of which regular polygons can be con-

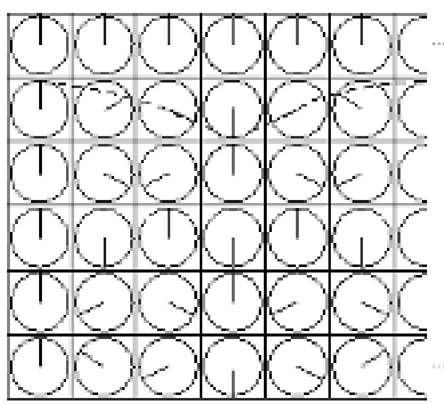

Figure 1 Phase operators: $\mathbf{Z}/6\mathbf{Z}$ discretization

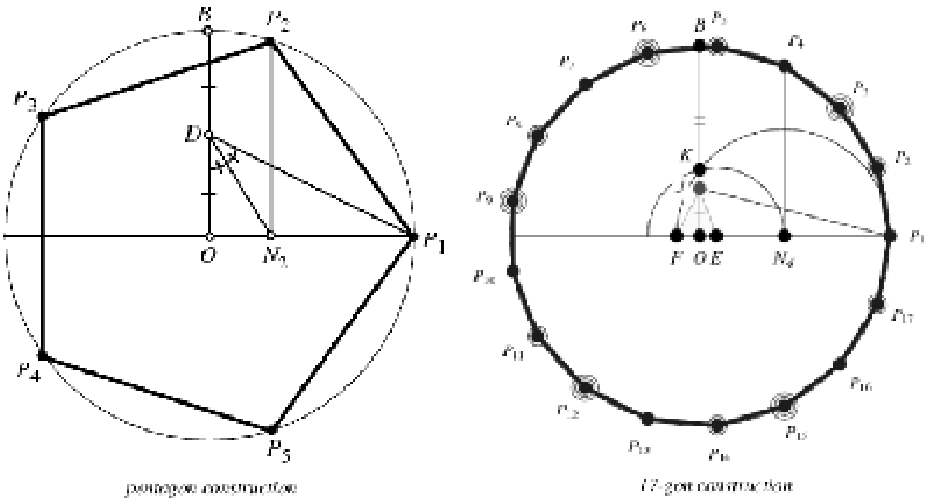

Figure 2 Construction of polygons by ruler and compass

structed using only ruler and compass (see Figure 2).

The crucial feature that allows a solution of this geometric problem is the fact that, in addition to the obvious rotational symmetries of regular polygons, there exists another hidden and much more subtle symmetry coming from the Galois group Gal($\bar{\mathbf{Q}}/\mathbf{Q}$), a very beautiful and still mysterious object, which in this case manifests itself not through the multiplicative action of roots of unity (rotations of the vertices of the polygons) but through the operation of raising them to powers.

Modular curves and non-commutative spaces

Thus, from the example of the Bost Connes noncommutative space, a dictionary emerges that relates the phenomena of spontaneous symmetry breaking in quantum statistical mechanics to the mathematics of Galois theory. Moreover, the partition function of this quantum statistical mechanical system is an object of central interest in number theory, namely the Riemann zeta function (see figure 6). More recently, other results that point to deep connections between noncommutative geometry and number theory appeared in the work of Connes and Moscovici on modular Hecke algebras, which showed that the Rankin–Cohen brackets, an important algebraic structure on modular forms extensively studied years ago by Don Zagier, have a natural interpretation in the language of noncommutative geometry. Modular forms are a class of functions of fundamental importance in many fields of mathematics, especially in number theory and arithmetic geometry. They exhibit elaborate symmetry patterns associated to certain tessellations of the hyperbolic plane (see figure 5).

Figure 3 An example of codimension one foliations

When viewed with the eyes of noncommutative geometry the algebraic structures studied by Zagier appear as a manifestation of a type of symmetry of noncommutative spaces, related to the transverse geometry of codimension one foliations (figure 3), which was investigated extensively in the work of Connes and Moscovici.

The special tessellations of the hyperbolic plane mentioned in relation to modular forms give rise to a family of 2-dimensional surfaces known as the modular curves. Recent work of Manin and Marcolli showed that much of the rich arithmetic structure of the modular curves is captured by a noncommutative space that arises from the tessellation restricted to the infinitely distant horizon of the hyperbolic plane (the bottom horizontal line in figure 5). The fact that the infinite horizon of modular curves hides a noncommutative space was also observed in the work of Connes, Douglas and Schwarz in the context of string theory.

Ongoing work of Connes and Marcolli uncovered the remarkable fact that all the instances listed above of interactions between number theory and noncommutative geometry (Connes' work on the Riemann zeta function, the Bost–Connes system, the modular Hecke algebra and the noncommutative boundary of modular curves) are, in fact, manifestations of the same underlying noncommutative space, namely the space of commensurability classes of **Q**-lattices.

Q-lattices

A lattice consists of arrays of points in a vector space, arranged like atoms in a crystal. For example, the set of points with integer coordinates in the plane is a 2-dimensional lattice. A **Q**-lattice is one such object where one has a way of labelling the points of rational coordinates inside the fundamental cell of the lattice. If each rational point is labelled in a unique way the **Q**-lattice is said to be invertible, while in general one also allows for labellings that miss certain arrays of points while assigning multiple labels to others (see figure 4).

When studying the geometric properties of **Q**-lattices, it is natural to treat as the same object all **Q**-lattices that have the same rational points and where the respective labellings agree whenever both are defined. This determines an equivalence relation on the set of **Q**-lattices. One observes that the identifications produced by this seemingly harmless equivalence relation are in fact drastic enough to give rise to a noncommutative space. On the other hand, if one restricts attention to just invertible **Q**-lattices, these are organized in a classical space. In the case of 2-dimensional lattices, the parameterizing space is the family of all modular curves.

Since **Q**-lattices exist in any dimension, there is in any dimension a corresponding noncommutative space. The Bost–Connes space is just the space of commensurability

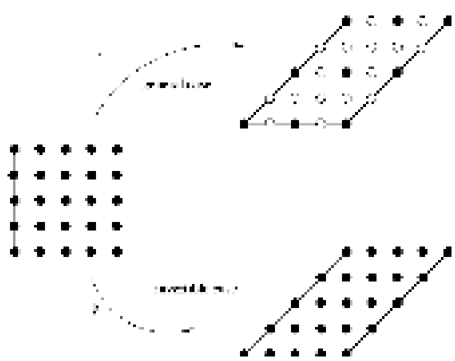

Figure 4 **Q**-lattices

classes of 1-dimensional **Q**-lattices considered up to a scaling factor. The noncommutative space introduced by Connes in the spectral realization of the zeros of the Riemann zeta function (whose position in the plane is the content of the Riemann hypothesis) is the space of commensurability classes of 1-dimensional **Q**-lattices with the scale factor taken into account. The modular Hecke algebra of Connes and Moscovici is a piece of the algebra of coordinates on the space of commensurability classes of 2-dimensional **Q**-lattices and the noncommutative boundary of modular curves is a stratum in this space that accounts for possible degenerations of the 2-dimensional lattice.

The noncommutative space of commensurability classes of 2-dimensional **Q**-lattices up to scale also has a natural time evolution and one can investigate the structure of the corresponding thermodynamic equilibria. At zero temperature this quantum space freezes on the underlying classical space (the family of modular curves) and all quantum fluctuations cease. The extremal states at zero temperature correspond to points on a modular curve. When the temperature rises quantum effects become predominant and the system undergoes a first phase transition where all the different equilibrium states merge, leaving a unique chaotic state. There is then a second critical temperature where the system experiences another phase transition after which no equilibrium state survives.

What acts as a group of symmetries of this quantum mechanical system is the group of all arithmetic symmetries of the modular functions. As in the 1-dimensional case, the induced action on the extremal states at zero temperature is via Galois theory. In this 2-dimensional system, however, not all symmetries act directly on the classical space at zero temperature as they need the more refined structure of the quantum system. Hence one obtains the Galois action at zero temperature by warming up below critical temperature, looking at the full symmetries of the quantum system, and then cooling down again to zero temperature where arithmeticity becomes apparent.

Zeros of the Riemann zeta function

The noncommutative space of commensurability classes of **Q**-lattices with its rich arithmetic structure provides a valuable tool for investigating many related number theoretic questions. For instance, in the spectral realization of zeros of the Riemann zeta function an important question is how to pass consistently to extensions of the field of rational numbers. In the case of imaginary quadratic fields (extensions $\mathbf{Q}(\sqrt{-d})$ of the rational numbers by an imaginary number that is the square root of a negative integer) an analogue of the Bost–Connes quantum statistical mechanical system that has the same properties and the same relation to the Galois theory of abelian extensions was constructed in more recent work of Connes, Marcolli and Ramachandran. The Galois theory of abelian extensions of imaginary quadratic fields is related to the beautiful theory of elliptic curves with complex multiplication and in fact the corresponding quantum statistical mechanical system has a natural formulation in terms of the Tate modules of elliptic curves and the isogeny relation. It can be seen as a specialization of the dynamical system of Connes–Marcolli for **Q**-lattices of rank two, when restricted to those **Q**-lattices that are also 1-dimensional lattices over the field $\mathbf{Q}(\sqrt{-d})$. The construction of Connes–Marcolli was further generalized by Eugene Ha and Frédéric Paugam to a large class of interesting moduli spaces in arithmetic geometry: Shimura varieties, with the Bost–Connes and the Connes–Marcolli systems representing the simplest zero-dimensional and 1-dimensional cases. Benoit Jacob and, using different methods, Consani and Marcolli extended the Bost–Connes construction further to the positive characteristic case of function fields of curves over finite fields. Instead of **Q**-lattices and commensurability, one works in this case, similarly, with Tate modules of rank one Drinfeld modules and isogeny.

An especially interesting and challenging case is that of real quadratic fields $\mathbf{Q}(\sqrt{d})$. Understanding the Galois theory of abelian extensions of such fields is a very important open problem in number theory. A main obstacle comes from the fact that one is missing geometric objects playing in this case the same role that elliptic curves with complex multiplication play in the case of imaginary quadratic fields. In an inspiring and groundbreaking paper, Yuri Manin outlined a striking parallel between the theory of elliptic curves with complex multiplication and the theory of noncommutative tori with real multiplication. This suggests that noncommutative geometry may well provide the missing structure that is needed in this case. There are many challenges implicit in implementing this 'Real Multiplication Program', most importantly the fact that one needs to identify suitable 'coordinate functions for torsion points' on the real multiplication noncommutative tori, analogous to the role that the Weierstrass $\wp$-function plays for elliptic curves. Re-phrased in terms of the quantum statistical mechanical systems of Bost–Connes type, this problem consists of identifying the 'arithmetic elements' in the algebra of the quantum statistical mechanical system associated to the real quadratic field by the general Ha–Paugam construction. Working directly with the noncommu-

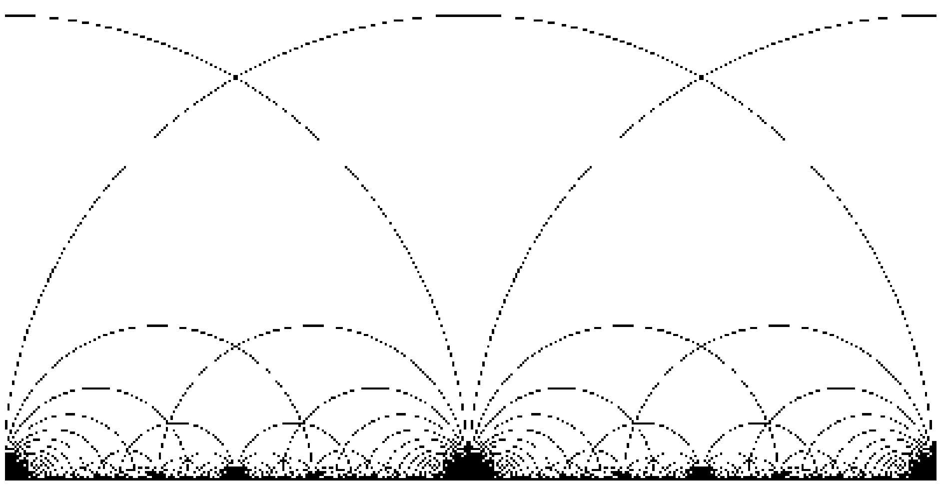

Figure 5 The modular curve

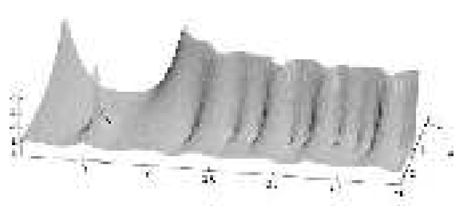

Figure 6 Absolute value of the Riemann zeta function

tative tori, several important advances have been made in the past couple of years, starting with a very important contribution by Polishchuk, who showed that the noncommutative tori with real multiplication have an algebraic model as an algebro-geometric noncommutative space, in addition to their usual analytic model. This algebraic version was related to Manin's quantized theta functions by Marya Vlasenko, while an explicit presentation in terms of modular forms was given by Jorge Plazas. The same algebraic model of noncommutative tori developed by Polishchuk and Polishchuk–Schwarz also provided the basis for a Riemann–Hilbert correspondence for noncommutative tori of Mahanta–van Suijlekom. The analytic model of noncommutative tori can also be used to obtain arithmetic information: Marcolli recently showed that the Shimizu L-function of a lattice in a real quadratic field is naturally obtained from a Lorentzian metric (spectral triple) on the noncommutative torus. The 'Real Multiplication Program' remains a rapidly developing and exciting part of the interaction between noncommutative geometry and number theory.

The Weil proof of the Riemann hypothesis for function fields

Connes' approach to the Riemann hypothesis featured prominently in recent developments in the field, through the work of Connes–Consani–Marcolli on *endomotives* and spectral realizations of L-functions. Abstracting from the class of examples of Bost–Connes-like quantum statistical mechanical systems mentioned above, one can identify a pseudo-abelian category of noncommutative spaces that combines the simplest category of motives, the Artin motives of zero dimensional algebraic varieties, with actions by endomorphisms of abelian semigroups. The algebra of the Bost–Connes system can be seen as an example of a semigroup action on a projective limit of Artin motives, and one obtains a large class of similar examples from self maps of algebraic varieties. These noncommutative spaces have a natural time evolution, induced from a counting measure on the algebraic points of the zero-dimensional varieties, and one can study the associated thermodynamic equilibrium states at varying temperatures. The low temperature equilibrium states provide a good notion of *classical points* of a noncommutative space and the restriction map at the level of algebras that corresponds to the inclusion of the classical points is defined as a morphism in an abelian category of 'noncommutative motives'. The cokernel of this map and its cyclic homology carry a scaling action of the positive real numbers, which is related to the ambiguity in choosing a Hamiltonian for the time evolution. The scaling action on the cyclic homology provide an analogue of the Frobenius action in the characteristic zero case. In the work of Consani–Marcolli on function fields it is shown that this same scaling action in the function field case is indeed obtained from the action of Frobenius (up to a Wick rotation to 'imaginary time'). The work of Connes–Consani–Marcolli shows that the scaling action on the cyclic homology of the cokernel of the restriction map gives a spectral realization of the zeros of the Riemann zeta function (or of L-functions with Grössencharakter) and that a cohomological version of Connes' result on the Weil explicit formula as a trace formula holds on this same cyclic homology. In this formulation the Riemann Hypothesis becomes equivalent to a positivity problem for the trace of certain correspondences on the underlying noncommutative space. This leads to a very suggestive dictionary of analogies between the Weil proof of the Riemann Hypothesis for function fields, which is based on the algebraic geometry of the underlying curve over a finite field, and the noncommutative geometry notions involved in the Connes trace formula. Expanding and developing this dictionary of analogies is another current focus of research in the field.

Acknowledgement
This text is reworked from a german original in Jahrbuch der Max-Planck-Gesellschaft 2004, pp. 395–400, www.mpg.de/bilderBerichteDokumente /dokumentation /jahrbuch /2004 /mathematik/for schungsSchwerpunkt/pdf.pdf (used with permission).

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Research

Computing Kazhdan-Lusztig-Vogan polynomials for split E_8

In 2007, Lie group theorists suddenly became the center of the media's attention when it was announced that the structure of the most complicated of the exceptional simple Lie groups, E_8, had finally been mapped. The precise result was in fact much more complicated, and might appeal less to the media: after intensive computer calculations, the Kazhdan-Lusztig-Vogan polynomials for the split real Lie group of type E_8 had been determined completely. The tale behind this computer algebra project is told by Marc van Leeuwen, professor at the Université de Poitiers, and one of the contributors to the solution of this problem.

On 19 March 2007, the project *Atlas of Lie groups and representations* suddenly found itself at the centre of a media blitz surrounding the announcement of the computation of the Kazhdan-Lusztig-Vogan polynomials for the split real Lie group of type E_8, which produced many gigabytes of data that could 'cover Manhattan'. While there had been a deliberate decision by the American Institute of Mathematics to organize publicity around a result obtained some months before, which would otherwise certainly not have attracted much attention outside the project, the sudden interest the announcement generated surprised all those involved. As would be the case for most subjects in mathematical research, it was impossible to explain to a general public what exactly had been done or why this was of such interest; we had to accept reading gross descriptions, such as that we had succeeded in finally 'decoding' E_8. One can only conclude that there must be many who, without knowing the details or seeking to, love mathematics in general and E_8 in particular.

David Vogan has explained in the *Notices* of the American Mathematical Society the broader representation theoretic context in which the computations of the Atlas project take place and the one concerning E_8 in particular. In this article I will pursue a more modest goal of giving some indications of the nature of the computation and the mathematical objects involved.

Computational Lie theory and 'Atlas'

In Lie theory one studies groups that are also differentiable manifolds, a subject that at first sight seems not to offer the kind of possibilities for exact algebraic computation that exist for instance for finite groups. Yet the structure and representation theory of (reductive) Lie groups mainly employs derived combinatorial structures such as root systems rather than elements of the groups themselves and this in fact makes the subject very suited to a computational approach. Thus when sufficiently powerful computers became generally available around the 1980s, programs were developed for performing the essentially combinatorial computations occurring in this theory; a notable example is the program LiE, developed around 1990 in a project headed by Arjeh Cohen at the CWI in Amsterdam, which computes with representations of complex reductive Lie groups and which is still in use today. Kazhdan-Lusztig polynomials for Coxeter groups, which were first defined around 1980, are also of great importance in this theory, but although they are given by elementary recursion relations, their computation poses unique computational challenges that are best handled by a dedicated program. The most powerful such program *Coxeter* was written, and continually improved, during the 1990s by Fokko du Cloux, a Dutch mathematician living in Lyon.

In 2002, at a workshop on computational Lie theory held in Montreal, the first plans for a project that was to be named *Atlas of Lie groups and representations* were made by Jeff Adams, Dan Barbasch, Fokko du Cloux, John Stembridge, Peter Trapa and David Vogan. Their purpose was to apply computational methods also to the more complicated theory of real Lie groups. In particular the notion of Kazhdan-Lusztig polynomials can be generalised to this setting, although the 'parameter set' on which it depends is a considerably more complicated combinatorial structure than the Weyl group that is used in the case of complex groups. I became involved with this project when it was

already under way in 2003 through an invitation by Fokko (based on my experience in the LiE project) to participate in the considerable programming effort that this new project would require. After finishing his *Coxeter* program in early 2004, Fokko started working on this new program *atlas*. By the end of 2005 he had implemented all the necessary concepts from the structure theory of real reductive Lie groups and adapted the Kazhdan-Lusztig algorithms to this setting (while I was only starting to get to grips with the C++ programming language and some of the code he had written). At that point the tables of Kazhdan-Lusztig-Vogan (KLV) polynomials could be computed for all cases of interest, except for the big block of the split real form of E_8.

Classical and exceptional groups

To understand the special place of E_8 in computational Lie theory, let me say a few words about the classification of reductive Lie groups, simplifying somewhat to focus on the essential points. The first classification is that of *complex* reductive groups. These have a 'semisimple part' and a 'central torus'; the possible presence of the latter serves mostly to allow important examples like $\mathbf{GL}(n,\mathbf{C})$ (and to complicate certain algorithms) but is otherwise of minor importance. The semisimple part is essentially determined by its 'root system', which is a finite set of vectors in a finite (and relatively low) dimensional vector space, and it satisfies a number of symmetry conditions. These conditions are so restrictive that root systems can be completely classified, which was done by Killing and Cartan at the end of the 19th century. They are a product of 'simple' factors, each of which is either a member of one of four infinite families $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 2}$, $(C_n)_{n\geq 3}$ and $(D_n)_{n\geq 4}$, or one of the five types labelled G_2, F_4, E_6, E_7 and E_8. The members of the infinite families occur as root systems of certain 'classical' Lie groups such as the special linear and symplectic groups; the remaining ones correspond to 'exceptional' simple Lie groups.

The classical root systems have a very regular description valid in all possible dimensions; for instance, the root system of type D_n can be realized as the set of all vectors of length $\sqrt{2}$ in the sublattice $\mathbf{Z}^n$ of $\mathbf{R}^n$ equipped with the standard scalar product (the cases D_2 and D_3 are omitted from the classification only because they coincide with certain other root systems). The exceptional root systems, however, depend on particular circumstances arising only in specific dimensions. For instance, the given description realizes the 24-element root system of type D_4 by vectors like $(1,0,-1,0)$, but a similar copy of this system scaled by a factor $\sqrt{2}$ can be found by taking all vectors of length 2 in $\mathbf{Z}^4$, like $(0,2,0,0)$ or $(-1,1,-1,-1)$; the union of these two systems forms the 48-element root system of type F_4. The systems of type G_2 and E_8 can be similarly formed due to singular circumstances, while those of type E_6 and E_7 are most easily described as root subsystems of the type E_8 system. The latter system has the following description. Starting with the usual 112-element root system of type D_8 contained in $\mathbf{Z}^8$, one adds all vectors in the lattice $(\frac{1}{2}\mathbf{Z})^8$ that (also) have length $\sqrt{2}$ and an even sum of coordinates, like $(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$; this adds 128 vectors to make a total of 240 vectors for the type E_8 root system. This collection of vectors has an exceptionally high degree of symmetry (in which the members of the original D_8 subsystem are in no way distinguished from the others), as witnessed by the fact that for each of the 240 roots, the reflection in the hyperplane orthogonal to it stabilises the set of all roots; these reflections generate a finite group of order 696729600, the Weyl group.

In computational Lie theory many questions can be broken down according to the simple factors, so most attention is on handling those cases. Among them, the exceptional types have a special place for several reasons. First, as their number is finite, one may hope to completely solve certain questions for exceptional types by computational means. On the other hand the regularity of classical types makes it possible that answers to some questions can be given by an abstract (combinatorial) description, like the Littlewood–Richardson rule that describes tensor product decompositions in all types A_n. Also, exceptional groups, and in particular E_8, appear to have a more dense and complicated structure than classical ones, making computational problems

Members of the Atlas project; the fourth person from the left is Fokko du Cloux.

more challenging for them; this was in particular the case for the Kazhdan-Lusztig calculation. Thus E_8 serves as a 'gold standard': to judge the effectiveness of the implementation of a general algorithm, one looks to how it performs for E_8.

In the Atlas project the interest is in real Lie groups, which means that a 'real form' needs to be specified in a complex group. In type E_8 there are three inequivalent real forms. The mentioned parameter set for KLV polynomials, a combinatorial structure called a 'block', depends in a complicated way on the real form and on a real form in the dual complex group. Much of the code of the *atlas* program serves to construct blocks. The complex group E_8 happens to be its own dual, so there are 9 possible combinations of real form and dual real form; the *blocksizes* command of *atlas* tells how big the corresponding blocks are (some are empty):

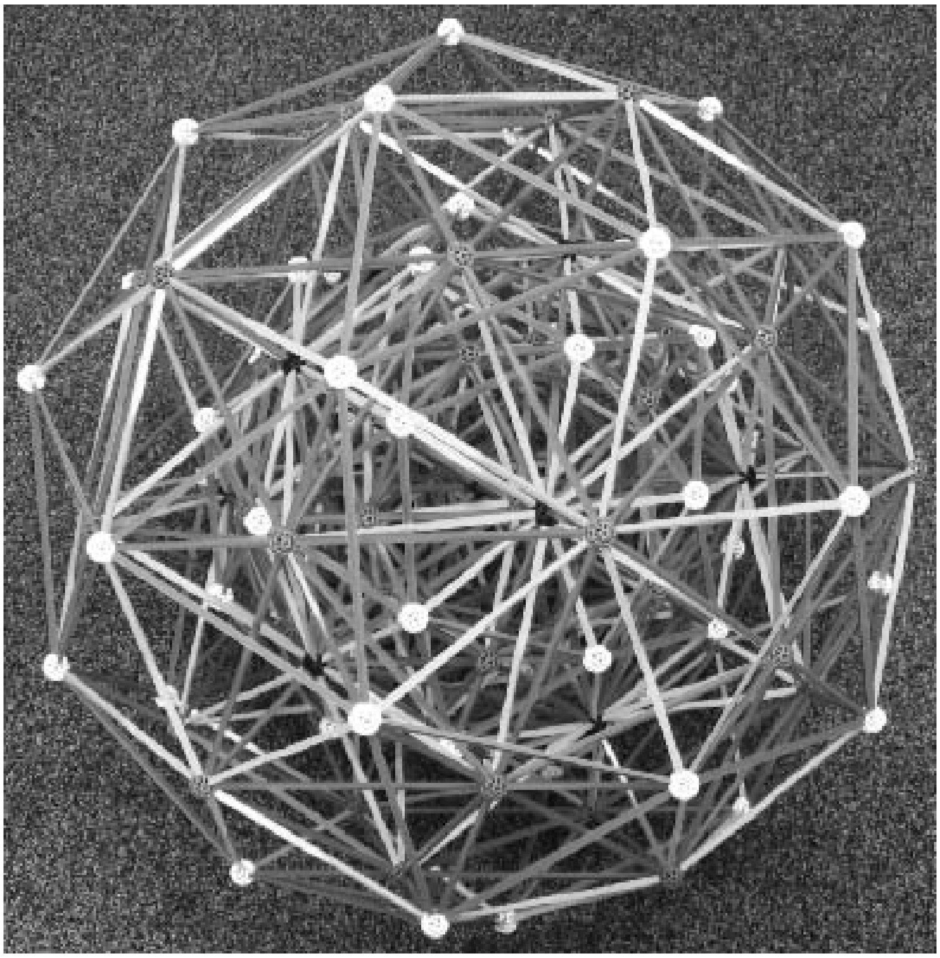
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 3150 & 73410 \\ 1 & 73410 & 453060 \end{pmatrix}$$

So the first real form has just one block, with one element (this happens for the *compact* real form in every type), the second has two substantially larger blocks and the last block of the third (split) real form is referred to as the 'big block'. Since KLV polynomials depend on *two* parameters ranging over a block, the particular interest in the size of the big block can be imagined, a size that was not known before the *atlas* program was written. It turns out to be quite small compared to the Weyl group; but still, 453060^2 is approximately 2.05×10^{11}.

The challenge of the big block of E_8

Block elements correspond to classes of irreducible representations of the real Lie group considered, and KLV polynomials provide intricate structural information that applies to this class. For instance, the irreducible representations of a *compact* group, all finite dimensional, form a single class. The existence of a single KLV polynomial for them (in fact the constant polynomial 1) means that they are all governed by a single formula: Weyl's character formula (on which much of LiE's computations are based).

A KLV polynomial is an element of $\mathbf{Z}[q]$ whose coefficients are known to be non-negative. The theory provides as the only means of computing them a collection of relations, expressed in terms of the combinatorial



Source: http://upload.wikimedia.org/wikipedia/en/2/2b/E8.roots.zome.jpg

Figure 1 David A. Richter's *Zome* three dimensional model of the rootsystem of the E_8. See his website for building instructions and many other interesting models.

structure of the block, which can be used as recursion relations to successively compute these polynomials for all pairs of parameters ranging over a given block. This collection of polynomials is naturally thought of as a matrix, which is lower triangular, and recursion relations then refer to entries both in rows above and in columns to the right of the current entry. The nature of these relations makes it pointless to try to compute individual KLV polynomials or to compute them without storing the computed values for later retrieval. Thus *atlas* basically just fills this matrix of polynomials according to the recursion relations and then writes all results to a file.

As long as available memory suffices to store all the results, this computation is quite fast; this explains why all interesting cases other than the big block for E_8 could be handled fairly easily once the program was written. Even computing the KLV polynomials for the block of size 73410 for the middle real form of E_8 takes less than 10 minutes on a modern PC, provided it is equipped with at least 2 gigabytes (GB) of memory. This particular case can be handled because not *all* entries in the triangular part of the matrix are stored; that would amount to some 2.7×10^9 entries, certainly requiring much more than 10 GB. Any entry that some recursion relation simply equates to a previously computed one is not stored (it will be computed on the fly should its value be needed), nor is any entry that turns out to be zero. In the mentioned case this means that only 63 million entries, which amounts to 2.5% of the triangular region, are actually stored.

However, when memory runs out the situation deteriorates rapidly. By using virtual memory, one can avoid simply having to abandon the computation: part of the storage may be written to disk (to so-called swap space) to create space for new values. But although the operating system attempts to keep in memory the most recently used values, it turns out that this set shifts constantly so that quite soon the computation starts spending nearly all of its time swapping (exchanging values between memory and disk). This is what happened for the big block of E_8: even with several gigabytes of memory and practically unlimited swap space, *atlas* was spending weeks just exercising the disk, while not even coming near to complete computation of the matrix of polynomials.

Meanwhile, we were shocked to learn that Fokko had been diagnosed with the neurodegenerative disease ALS, just around the time he finished programming the KLV algorithm; within months he lost the use of his hands and he died within a year. Though unable to program himself, he continued working on the Atlas project throughout that year, helping David Vogan, Jeff Adams and myself to get to grips with his program and to improve it further. Actually being able to do the big block of E_8 was not the primary concern for Fokko, as he told us that in a few year's time we would all have computers with enough memory to do it. He might very well turn out to have been right but in the end we did not have as much patience as he did.

Cracking E_8

We had contacted William Stein of Washington State University, who gave us permission to run *atlas* on a machine *sage* that had 64 GB of memory. However, it seemed that we would need a machine with at least 150 GB of memory to do the E_8 computation. While we considered buying an even larger computer, an idea occurred that triggered an attempt to adapt *atlas* so that it could complete the E_8 computation on *sage*. Two characteristics distinguishing the big block of E_8 from simpler cases are that the number of *distinct* polynomials in the matrix grows relatively fast (many polynomials occur extremely often so *atlas* stores only one copy of each) and that some polynomials require rather large coefficients (larger than $2^{16} = 65536$), which meant we needed to reserve 4 bytes for each coefficient. Combined, this means that a lot of storage is needed just to record all the polynomials occurring: it involves more than 13 billion 4-byte coefficients. The idea was that the form of the recursion relations allowed the computation to be performed multiple times with modular coefficients, each of which can be stored in a single byte, and that the Chinese remainder theorem could be used afterwards to reconstruct the actual integer coefficients.

As the *atlas* program is well-structured, it was quite easy to modify it so that storage for polynomial coefficients was reduced to one byte and that all arithmetic on them was replaced by modular arithmetic. This modified program was ready in early December 2007, just a few days after the idea of using modular arithmetic had been adopted (and less than a month after Fokko's death). However, this modification, even though it divided the storage requirements for coefficients by 4, did not suffice to be able to do the big block even on *sage*, as other important factors of storage requirement were unaffected, i.e. the matrix telling which polynomial occurs where, the data structure used to allow each polynomial to be represented just once and the general overhead of data structures. Normally one trusts the libraries used to provide convenient data structures and does not bother to fiddle with details of bits and bytes; however, with data types that occur billions of times and memory at a premium, it pays to push for improvement in these areas. In fact it turned out that much could be gained with fairly straightforward techniques like replacing 8-byte pointers with 4-byte identification numbers, using a hash table rather than a search tree and gathering all polynomial coefficients in a single array rather than in a billion different ones. I had been familiar with such methods since a long time, having studied programs (like TeX) written by Donald Knuth at a time when computer memory was less abundant than it is today. In the end, the memory requirement for the big block was brought down to about 55 GB, making it feasible to try it on *sage*.

Everything, including some small utility programs to perform lifting of modular coefficients, was more or less ready to run at the start of the Christmas holidays. The computations were actually run in Seattle via remote control by David Vogan, from MIT or whatever other place with Internet connection (like an airport lounge) he happened to be at, and with other Atlas members 'looking over his shoulder' through VNC windows. I will not repeat the beautiful description in David's article of our varying fortunes as the computations progressed. Suffice it to say that every first attempt at a part of the computation failed, either because of a programming error that previous testing had failed to unveil or due to one of many system crashes (which turned out to be caused by independent processes on *sage*). Nevertheless, with obstinacy we managed to push on to the finish line on 8 January 2008.

My personal recollection of this adventure is one of some short interruptions of the holidays spent at my home in Poitiers with my two teenage children, in which I discussed software problems with David by email (he actually found and repaired most of them himself), and of a final weekend back in the Netherlands where, after having returned the children to their home, I spent an evening at my parents' pondering over a last bug that David had signalled. Early the next morning, having found that an overnight trial run had failed to reproduce any error, I wrote an email back to give David the go-ahead for another try (because the bug had accidentally been fixed already!), while my father was worrying that I was going to miss my train back to France (which I didn't). For once, computers needed more time than trains: I was already back to work the next day before the final run of the Chinese remainder program completed.

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How complicated can structures be?

Is there a measure of how 'close' non-isomorphic mathematical structures are? Jouko Väänänen, professor of logic at the Universities of Amsterdam and Helsinki, shows how contemporary logic, in particular set theory and model theory, provides a vehicle for a meaningful discussion of this question. As the journey proceeds, we accelerate to higher and higher cardinalities, so fasten your seatbelts.

By a *structure* we mean a set endowed with a finite number of relations, functions and constants. Examples of structures are groups, fields, ordered sets and graphs. Such structures can have great complexity and indeed this is a good reason to concentrate on the less complicated ones and to try to make some sense of them. In this article we walk the less obvious and perhaps less appealing trail of delving more deeply into more and more complicated structures. We raise the question of how we can make sense of the statement that we have found an extremely complicated structure? This is typical of the kind of question investigated in mathematical logic. The guiding result of mathematical logic is the Incompleteness Theorem of Gödel, which says that the logical structure of number theory is so complicated that it cannot be effectively axiomatized in its entirety. In other words, the theory is non-recursive, i.e. there is no Turing machine that could tell whether a sentence of number theory is true or not. A contrasting and pivotal result of logic from the same period is Alfred Tarski's result that the field of real numbers (or the field of complex numbers) can be completely and effectively axiomatized and is indeed recursive in the sense that there is a Turing machine that decides whether a given statement about the plus and times of real (or complex) numbers is true or not.

We start with the extremely interesting situation concerning attempts to classify finite models. We then move to the more established case of countable structures. Sweeping results exist here and this case is very much the focus of current research. Then we turn our faces to the wind and stare into the eyes of the difficult uncountable structures. New ideas are needed here and a lot of work lies ahead. Finally we tie the uncountable case to stability theory, a recent trend in model theory. It turns out that stability theory and the topological approach proposed here give similar suggestions as to what is complicated and what is not.

For unexplained set theoretical concepts refer to [5].

Finite structures

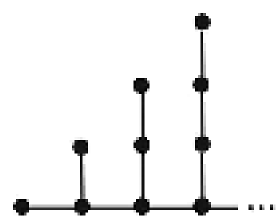
Let us start with finite structures. The famous P=NP question, one of the Clay Institute Millennium Questions, asks if we can decide in polynomial time whether a given finite graph is 3-colourable. Should the answer to the P=NP question be negative, as is expected, we will have a sequence of some rather complicated graphs, for which no algorithm, running in polynomial time in the size of the graph, can decide whether the graph is 3-colourable or not.

The problem of whether the isomorphism of two finite structures can be solved in polynomial time is a famous open problem of complexity theory. It is particularly famous because it is not known whether it is NP-complete either; it may be strictly between P and NP.

Countable structures

What about countably infinite structures? We should not distinguish between isomorphic structures. So let us assume the universe of our countable structures is the set $\mathbb{N}$ of natural numbers. After a little bit of coding, such countable structures can be thought of as points in the topological space N of all functions $f : \mathbb{N} \to \mathbb{N}$ endowed with the topology of pointwise convergence, where $\mathbb{N}$ is given the discrete topology.

We can now consider the orbit of an arbitrary countable structure under all permutations of $\mathbb{N}$ and ask how complex this set is in the topological space N. If the orbit is a closed set in this topology, we should think of the structure as an uncomplicated one. This is because the orbit being closed means, in view of the definition of the topology, that the finite parts of the structure completely determine the whole structure, as is easily seen to be the case in the graph of the picture:



A structure may be quite innocuous even if the orbit is not closed. For example, the orbit of the ordered set of the rationals is not closed because as far as the finite parts are concerned it cannot be distinguished from the order type of the integers. While not closed, the orbit of the rationals is of the form

$$\bigcap_n \bigcup_m F_{n,m}, \tag{1}$$

where each $F_{n,m}$ is closed. This is a consequence of the fact that the density of the order, as well as not having endpoints, can be expressed in the form 'for all ... exists ...', and these two properties completely determine the structure among countable structures.

When the number of alternating intersections and unions increases in the formula (1), even to the transfinite, we end up with the hierarchy of Borel sets

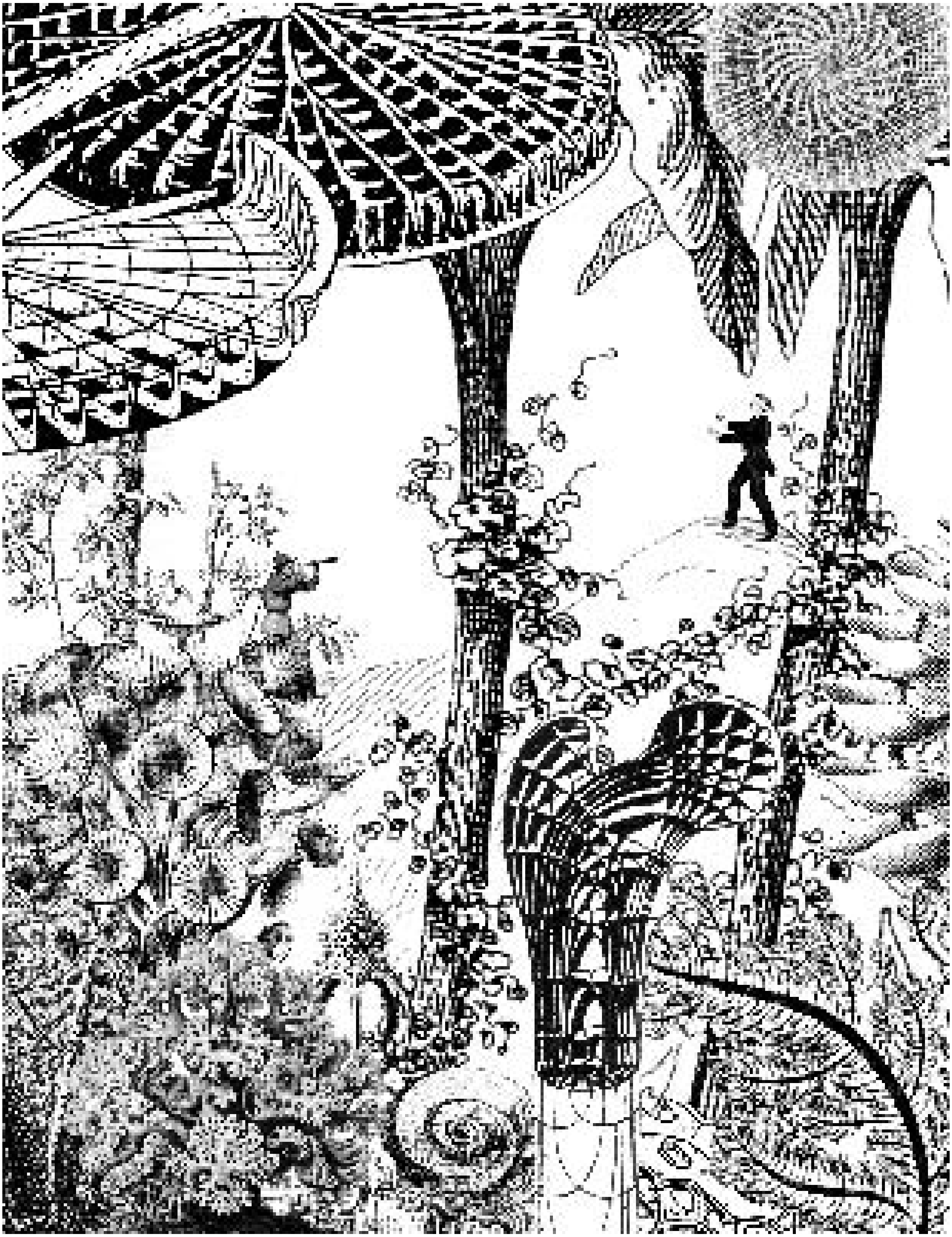


Illustration: Ryu Tajiri

- G_0 = open sets, F_0 = closed sets
- $G_{\alpha+1}$ = countable unions of sets from F_α
- $F_{\alpha+1}$ = countable intersections of sets from G_α
- $G_\nu = \bigcup_{\alpha<\nu} G_\alpha$, $F_\nu = \bigcup_{\alpha<\nu} F_\alpha$, if ν is a limit ordinal

named after Émile Borel (1871–1956), a French mathematician.

So the philosophy is now that the further the orbit is from being a closed set the more complicated the structure is. We can go up the Borel hierarchy and find structures on all levels $F_\alpha \cup G_\alpha$. By a deep result of Dana Scott [9] every orbit is on some level of the Borel hierarchy, although *a priori* the orbits are just analytic sets, i.e. continuous images of closed sets.

The levels $F_\alpha \cup G_\alpha$ of the Borel hierarchy are calibrated by countable ordinals α. Orbits of familiar structures such as $(\mathbb{N}, +, \cdot, 0, 1)$, the field of rational numbers, the Random Graph, the free Abelian group of countably many generators, and any vector space (over $\mathbb{Q}$) of countable dimension are all on one of the lowest infinite levels of the Borel hierarchy. On the other hand the orbit of any sufficiently closed countable ordinal $(\alpha, <)$ is on level α, i.e. in the set F_α but not in any $F_\beta \cup G_\beta$ for $\beta < \alpha$ (what is needed is that α is such that $\beta < \alpha$ implies $\omega^\beta < \alpha$). Such structures of high level can be constructed for e.g. Abelian groups. This basic setup has led recently to a rich theory of Borel equivalence relations on Polish spaces [1].

Uncountable structures

What if we have an uncountable structure and we want to measure its complexity and the degree to which a given structure is close to being isomorphic to it? After all, the most important mathematical structures, such as the fields of real numbers and complex numbers, Euclidean spaces, Banach spaces, etc., are all uncountable. In the light of our experience with countable structures, it seems natural to consider structures that are determined by their countable parts as uncomplicated.

Consider the ordered set $L = (\mathbb{R}, <)$ of all real numbers. If we only look at countable sub-orders, this is no different from the ordered set $L' = (\mathbb{R} \setminus \{0\}, <)$ of the non-zero real numbers — although L and L' are not isomorphic as the first is a complete order and the second is not. In fact, L is quite a complicated structure albeit not by any means among the most complicated. One example of the peculiar properties of L is the following. If we add a new real to the universe by Cohen's method of forcing, L becomes isomorphic with L'. So in some sense L is a hair's breadth away from being L'. The fact that L is complicated is related to exactly this kind of phenomenon, to being an iota away from another, non-isomorphic structure.

When we set our foot on the path of looking at uncountable structures through the lens of their countable parts, the first resting spot is bound to be the class of structures that can be expressed as an increasing union of countable substructures or, equivalently, structures of cardinality $\aleph_1$, the first uncountable cardinal. Now the alarm bells start to ring! We do not know whether the real numbers, the complex numbers, Euclidean space, Banach spaces, etc, have this property.

The question of whether the set $\mathbb{R}$ of real numbers is an increasing union of countable sets is known as the Continuum Hypothesis (CH). So in order to include those structures in this discussion we have to assume CH. In fact, most of the currently known results in this direction assume CH anyway. But there is a whole family of structures that are by their very definition increasing unions of countable structures, and this family is closed under various algebraic operations but not under infinite products, unless we assume CH. An example is the order-type $(\omega_1, <)$ and the numerous structures built around it, such as the free Abelian group on $\aleph_1$ generators.

Models of cardinality $\aleph_1$ can be thought of as points in the space N_1 of functions $f : \omega_1 \to \omega_1$ endowed with the topology of pointwise convergence, that is, a neighbourhood of a point $f \in N_1$ is of the form $N(f, X) = \{g \in N_1 : \forall x \in X(g(x) = f(x))\}$ where X is countable. So the orbit of a structure is a closed set essentially if the countable parts of the structure completely determine it. The orbit of the free abelian groep $F(\aleph_1)$ on $\aleph_1$ is not at all closed. There are so-called almost free Abelian groups, every countable subgroup of which is free but which are not free themselves. So the question to ask is not only what the countable substructures are but also how they sit inside the structure. To see how complicated the group $F(\aleph_1)$ is let us define the Borel hierarchy in N_1.

The class of *Borel* sets of the space N_1 is the smallest class of sets containing the open sets and closed under complements and unions of length ω_1. A set is *analytic* if it is a continuous image of a closed subset of N_1. Orbits of structures of cardinality $\aleph_1$ are, *a priori*, analytic, but are they Borel?

When we carry out the same topological analysis of models of cardinality $\aleph_1$, as we did with countable models, the notion of an approximation is more complex. After all, we have to approximate an uncountable object and we cannot approximate $f \in N_1$ merely by its finite initial segments. Roughly for the same reason, the approximations are scaled by bounded trees, i.e. trees with no uncountable branches, rather than ordinals.

The passage from well-founded trees to bounded trees brings with it two major problems. The first is the problem of the ordering of the class of all such trees. The problem of the ordering of the trees is the following. In analogy with ordinals and well-founded trees we write $T \leq T'$ if there is a strict tree-order preserving (but not necessarily one-to-one) mapping from T to T'. For well-founded trees this quasi-order is connected, i.e. any two well-founded trees are comparable by $\leq$. For non-well-founded trees this need not be the case [4, 13]. This is on the one hand a shortcoming of the whole approach, as it means that some structures are incomparable as to their complexity. On the other hand it has un-

Émile Borel (1871–1956), a French mathematician and politician who has many theorems named after him; there is even a Borel crater on the moon in the *Mare Serenitatis*. Borel, together with Lebesgue and Baire, is also known as a representative of *semi-intuitionism*, an alternative approach to constructivism along with Brouwer's *intuitionism*. The former maintained that set theory should be limited to definable sets, anticipating *descriptive set theory*, while the latter launched a criticism of the Law of Excluded Middle, leading to modern *intuitionistic logic* and *constructive mathematics*.

earthed a rich theory of trees, and whatever progress we can make in this direction is directly reflected in our ability to measure how close uncountable structures can be to each other in the given topology. One puzzle that has arisen in this connection is the existence of a *Canary tree*. This name is due to the following special role of Canary trees. If any stationary subset of ω_1 is killed by forcing without adding reals, then the Canary tree gets a long branch. Summing up, if a stationary set is poisoned somewhere then the Canary tree warns us by expiring. The existence of Canary trees cannot be decided on the basis of ZFC or even CH alone [7]. However, Canary trees are intimately related to the complexity of some canonical structures. It can be shown that there is a Canary tree if and only if the orbit of the free Abelian group of $\aleph_1$ generators is analytic co-analytic [6]. In the space N, analytic co-analytic sets are Borel but in the space N_1, the situation is more complicated. The most promising attempt to bring order into the chaos of bounded trees is the approach of Todorcevic [12] under the assumption of the so-called Proper Forcing Axiom (PFA). This axiom says, very roughly speaking, that the universe is invariant under changes imposed by a certain restricted form of Cohen's concept of forcing. Another approach is to restrict to sufficiently definable trees and thereby avoid the incomparability problem [2].

Another new feature that arises in the study of N_1 is the fact that if we assume CH, the Luzin Separation Principle fails. There are disjoint analytic sets that cannot be separated by a Borel set [11]. This further emphasizes how the difference between the countable and the uncountable is reflected in the topology of N and N_1, and thereby in the classification of countable versus uncountable models.

Stability theory

In modern model theory there is an alternative approach to the problem of classifying structures, namely stability theory [8]. The difference is that stability theory tries to classify complete first order theories rather than structures. However, the message of stability theory is that all models of size $\aleph_1$ of theories satisfying a combination of certain stability conditions (superstable, NDOP, DOTOP) are rather 'uncomplicated' in the sense that their isomorphism can be expressed in terms of a determined game (called the Ehrenfeucht-Fraïssé-game) of length ω [10]. Naturally, such structures may be very complicated in other ways. The point of stability theory is that in such structures one can define a kind of geometry that enables one to classify the structure in terms of dimension-like invariants. On the other hand, theories failing to satisfy such stability conditions are bound to have models of cardinality $\aleph_1$ that are extremely complicated. This is Shelah's 'Main Gap' [10]. For example, assuming CH and if there are no Canary trees, such theories have models of cardinality $\aleph_1$ with high complexity in the definability theoretic sense described in this article [3]. To measure the height of the complexity one uses bounded trees and it turns out that under the stated assumptions one can go beyond any bounded tree. Work in this direction is very much underway.

Conclusion

The study of the complexity of uncountable structures is an interdisciplinary subject. We need to develop set theory, and especially the theory of trees, in order to have a good measure of the complexity of uncountable models. At the same time, we have to develop model theory, and especially stability theory, in order to distinguish important dividing lines between simple and complicated structures. Both set theory and model theory suggest that we should look for an answer in the direction of long games. In model theory the relevant games are known as Ehrenfeucht-Fraïssé games. In set theory the corresponding games are related to the so-called stationary sets. The results referred to above connect the two games and thereby tie a knot connecting set theory and model theory. When we look into the deep eyes of the uncountable structures, we are perhaps starting to see there some compassion for our modest advances, our budding infinite trees, our courageous appeals to stability and our resolve to play the game to the end.

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History

Ancient and modern secrets of Isfahan

The art of medieval Islamic symmetry is well known throughout the world, albeit for a large part only from pictures. Recently, students from Leiden and Utrecht had the opportunity to see some extraordinary tilings through their own eyes on a study trip to Isfahan in Iran, under the guidance of Jan Hogendijk. As a result of this trip, a lasting contact was formed between the students here and those in Iran. Jan Hogendijk is professor in the history of mathematics in Utrecht and Leiden, and is a specialist on islamic mathematics.

On 27 December 2007, 39 mathematics and physics students from the Utrecht and Leiden Universities went with me on a study visit to Iran. The program had to include an extensive tour of Isfahan with its many spectacular Islamic tilings. Most of the tilings are based on pentagons and decagons but some involve heptagons and nonagons and so they cannot be constructed, at least not exactly, by ruler and compass.

An 11th century example is the tiling in figure 1, which is high on a wall of the North Cupola of the Friday Mosque. The tiling is also drawn in a medieval Persian manuscript in Paris. To illustrate the relationship, the drawing in the manuscript and three mirror images are superimposed on the tiling. The tiling arises from the tessellation of the plane by means of equilateral hexagons of two types (see the continuous lines in figure 2): type A with angles $4\alpha, 5\alpha, 5\alpha, 4\alpha, 5\alpha$ and 5α and type B with angles $4\alpha, 4\alpha, 6\alpha, 4\alpha, 4\alpha$ and 6α, where $\alpha = \frac{1}{7} \times 180^o$. The 'stars' inscribed in A and B have angular points at the midpoints of the sides of the hexagons and their angles at these points are 3α, 2α, 3α, 3α, 2α, 3α. The figures in bold broken lines in figure 2 are regular heptagons. When the North Cupola was constructed (in 1080 A.D.), the famous mathematician and poet Omar Khayyam lived in Isfahan, and some of his work is known to be related to other diagrams in the Persian manuscript. He may have invented this heptagonal tiling as well.

A more modern secret in Isfahan is its *House of Mathematics*, (www.mathhouse.org), which encourages mathematics awareness among highschool students. Teams of volunteers (consisting of highschool teachers and university students) work together with highschool students in groundbreaking educational projects. The circumstances are sometimes difficult but this only seems to make the staff more enthusiastic and more inventive. Dutch mathematics educators can learn a lot in Iran and formal cooperation agreements have been made between the House of Mathematics, the Freudenthal Institute at the University of Utrecht and Fontys teacher training college in Eindhoven. Our group of students was received warmly by the staff, who organized most of our scientific program in Iran.

Travelling to Iran is risky because of the danger of losing one's heart. Some students, who are showing the symptoms, are planning to organize a summer school on Islamic tilings in Isfahan in 2009, of course in cooperation with the House of Mathematics.

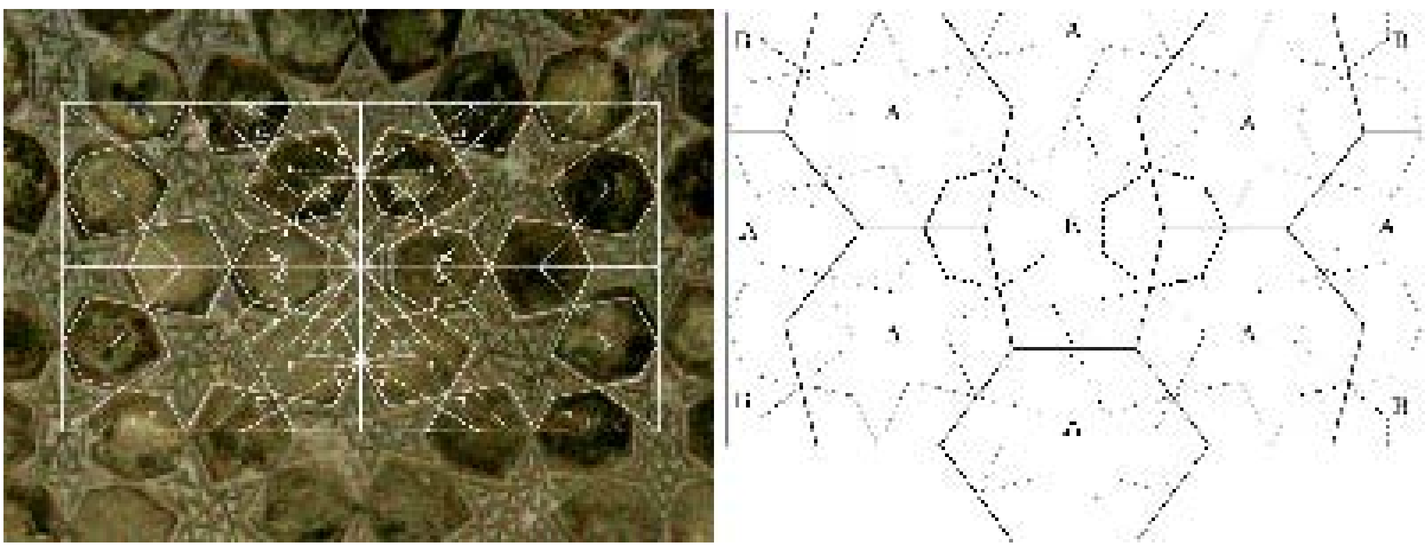

Figure 1 Left figure: Tiling in the North Cupola of the Friday Mosque. In white lines: drawing in ms. Paris, Bibliothèque Nationale, Persian Manuscript Ancien Fonds 169, f. 192a, with mirror images. Right figure: Drawing of the tiling. The left figure was produced by Mr Tom Goris, Eindhoven.

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Research

Three examples of applied and computational homology

Computational algebraic topology has already existed for some decades, with as its main objective the generation of examples. Nowadays, the field is rapidly changing into an applied branch of mathematics that is important in its own right. Robert Ghrist, topologist at the University of Illinois and one of the winners of the 2007 Scientific American 50 award, gives us three examples that illustrate this development, each with a different origin.

Mathematics is limitless in its dual capacity for abstraction and incarnation. To a large degree, many of the modern revolutions in technology and information rest on piers of mathematics that assist, inform, or otherwise catalyze progress. It appears that those branches of mathematics which are most easily understood and communicated are precisely those which find greatest applicability in the modern world. To conclude from this that deeper or more difficult fields are inherently less applicable would be premature.

Consider for example the utility of algebraic topology. Long cloistered behind formal and categorical walls, this branch of mathematics has been the source of little in the way of concrete applications, as compares with its more analytic or combinatorial cousins. In this author's opinion, this is not due to a fundamental lack of applicability so much as to

1. the lack of a motivating exposition of the tools for practitioners; and
2. an historical lack of emphasis on computational features of the theory.

These two issues are coupled. Advances which demonstrate the utility of a topological theory spur the need for good computation. Good algorithms for computing topological data spur the search for further applications.

Algebraic topology is the mathematics that arises in the attempt to describe the *global* features of a space via *local* data. That such tools have utility in applied problems concerning large data sets is not difficult to argue. To give a sense of what is possible, we sketch three recent examples of specific applications of homological tools. This list is neither inclusive nor ranked: these examples were chosen for concreteness, simplicity, and timeliness. This brief and woefully incomplete sketch is meant as an appetizer, for which the truncated bibliography serves as a menu for the second course.

On Homology.

Homology is a machine that converts local data about a space into global algebraic structure. In its simplest form, homology takes as its argument simple pieces of a topological space X and returns a sequence of *abelian groups* $H_k(X)$, $k \in \mathbf{N}$. Homology is a *functor*, which in practice means:

1. topologically equivalent spaces (homotopic) have algebraically equivalent (isomorphic) homology groups; and
2. topological maps between spaces

$$f : X \to Y$$

induce algebraic maps (homomorphisms) on homology groups

$$f_* : H_*(X) \to H_*(Y).$$

Numerous homology theories exist, fine-tuned for different classes of spaces (simplicial, cellular, singular, etc.).

Roughly speaking, homology groups count and collate *holes* in a space. The simplest example of a homological invariant is the number of connected components of a space — $\dim H_0$ — the type of 'holes' that a zero-dimensional instrument can measure. A less trivial example of a homological invariant is the *Euler characteristic*. The Euler characteristic χ of a triangulated surface is the alternating sum of the number of simplices — vertices minus edges plus faces — and that this quantity is a topological invariant of the surface. For more general (but tame) spaces, $\chi(X)$ can be expressed either as the alternating sum of the number of k-dimensional cells of X, or, as

$$\sum_{k=0}^{\infty}(-1)^k \dim H_k(X).$$

This quantity, being based on homology, is an invariant. It is a signal example of a homological device, being both computable and invariant. Our first example of applied algebraic topology relies on this invariant.

Looking Forward

The three examples here surveyed are all applications of homological tools to problems of large and often noisy data sets. However, there are numerous other examples of a different nature under the same aegis of applied algebraic topology. Many of these are *obstruction-theoretic* in nature — topological measures of complexity of coordinating robots, synchronizing a network, or performing distributed asynchronous computation.

The list of mathematical ideas which were once erroneously derided as useless abstractions (uniform convergence, matrix algebra, group theory, etc.) is sufficiently long and embarrassing so as to suggest patience in the case of applied algebraic topology. Given that the (hard) work of generating good algorithms for computing topological invariants for realistic systems is so recent [4], it can be successfully argued that the current spate of advances in applied algebraic topology is neither coincidental nor terminal.

How many people are in the building?

Problem: Target Enumeration. Consider a store whose ceiling tiles, walls, and carpet are embedding with people-counting sensors. How can these local sensors collaborate to determine the number of customers in the store?

Tool: Euler Characteristic Integration. One of the fundamental difficulties in large-scale sensor networks is *data aggregation*. A sufficiently dense collection of nodes will sample an environment redundantly. The goal of sensing is to compress this redundant local data into a global description of the environment. The operation of stitching local information over patches is the fundamental defining property of a *sheaf*, a means of assigning an algebraic object to open subsets of a space in such a manner that restrictions and overlaps are respected.

As an example, consider the problem of counting a collection of *targets*. Some fixed but unknown number N of targets lie in a domain D. The domain is filled with sensors, each of which can determine how many targets are nearby. It matters not how the sensors operate (*e.g.*, via infrared, acoustic, or optical sensing). Assume simple sensors which merely detect the number of nearby targets, with no information about target identity, distance, or bearing. In the continuum limit (where one has a sensor at each point in D), this yields a counting function $h : D \to \mathbf{N}$. The problem is to determine the number of targets, given only h.

The solution lies in an elegant integration theory which uses Euler characteristic as a *measure*. For compact sets A, B, the Euler characteristic satisfies $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. Note the similarity of this to the definition of a measure. Indeed, χ is a type of scale-invariant topological volume, as was known going back to Hadwiger and Blaschke at least. It is straightforward to construct a measure $d\chi$ against which one can integrate certain functions. The type of piecewise-constant or *constructible* function $h : D \to \mathbf{N}$ that a sensor field returns is integrable in this theory.

Recent work of Baryshnikov et al. [1] gives a simple formula for computing the number of targets as $\int_D h\, d\chi$, in the setting where each target is detected by sensors on a topologically trivial (*e.g.*, convex) neighborhood. Because this is a topological integration theory, there are no geometric restrictions. Sensors can, *e.g.*, count the number of vehicles driving over a domain laced with vibration sensors, counting subcompacts and SUV's as equals.

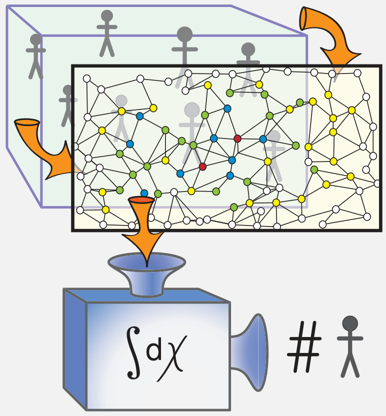

Figure 1 Integration with respect to Euler characteristic enumerates redundant data over a sensor network.

This is the starting point for a broad array of applications which rely on constructible sheaves and the sheaf-theoretic properties of $d\chi$. Precisely because the answer is expressed in terms of an integration theory, one can do the following:

- For a sparse network of sensor nodes, determining the number of targets becomes the numerical problem of approximating the topological integral via a discrete sampling.
- Thanks to a version of the *Fubini theorem* for $d\chi$, one can count moving targets over time without the need to embed clocks on the sensor nodes.
- Because integration is a local operation, target-counting can be performed by the network itself with a *distributed*, local computation.

Moral: "Data aggregation is a topological integration."

It looks chaotic to me!

Problem: Experimental Verification of Chaotic Dynamics. An experiment (physical or numerical) yields data that looks chaotic. Is it rigorously chaotic, or just noisy?

Tool: Conley Index Theory. One of the great scientific lessons of the 20^{th} century was that when a physical system exhibits erratic temporal behaviour, it may not be due to randomness or poor measurement — deterministic systems can exhibit well-defined *chaos*. However, it is a persistent challenge to demonstrate that a given system is chaotic. The *Lorenz equations* — themselves a cartoon model of fluid flow — were only recently shown to be rigorously chaotic, after more than thirty years' inquiry. Still more intractable remain data coming from physical experiments, in which system noise and instrument errors conspire to frustrate analysis. There seems to be little recourse for the experimentalist beyond saying: it looks chaotic to me.

A prime feature of topological methods is that, being global, they are typically impervious to the *noise* inherent in physical systems. Such is the case here. Work of Mischaikow et al. [5] uses a homological invariant of dynamics combined with *a priori* bounds on the noise amplitudes to determine the rigorous dynamics of an experimental system based on noisy time-series data.

The mathematical tool used is the *Conley index*, an algebraic-topological extension of the Morse index. Consider the flow of rainwater falling on a mountainous terrain D: this flow is that of $-\nabla h$, where $h : D \to \mathbf{R}$ is the height function of the terrain. The *Morse index* of a critical point of h is an integer that classifies the type of critical point: minima have index 0, saddle-passes have index 1, and maxima have index 2. The homological Conley index enriches the Morse index from integers to (homology types of) spaces: for a Morse function, the Conley index of a critical point is a *sphere* of dimension the Morse index.

The Conley index, unlike the Morse index, applied to non-gradient and non-smooth vector fields, as well as to discrete-time dynamics. It is efficacious, even to the point of detecting chaotic dynamics. This index is computable for realistic systems, thanks to recent progress in computational homology [4]. Work of Mischaikow et al. takes (noisy) *time-series* data and represents the dynamics as a *multi-valued map* on a cubical complex. By adapting the Conley index to this setting and computing the homological index, it is possible to verify the underlying dynamics, so long as the noise tolerances respect the discretization assumptions. Rigorous results about experimental or numerical data include the following:

- For experimentally-generated data on the dynamics of a *magneto-elastic ribbon* in an oscillating magnetic field, a Conley index approach proves that the experimental system is chaotic (has positive topological entropy) [5]. The method is robust, and works even when environmental noise alters the appearance of the data significantly.
- Numerical simulations of the *Kuramoto-Sivashinsky* partial differential equation indicate various stationary solutions. A Conley index computation [6] proves that these solutions exist, with a computational effort of the same order as a re-run of the numerical solution at a finer resolution.

Moral: "It's hardly more expensive to prove the dynamics than to simulate it."

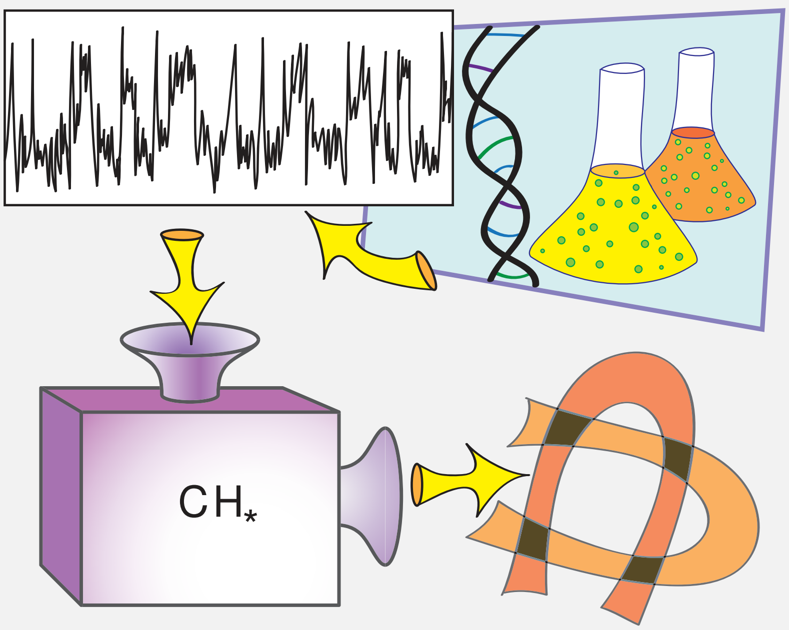

Figure 2 The Conley index CH_* of experimental time series data can rigorously verify chaotic dynamics.

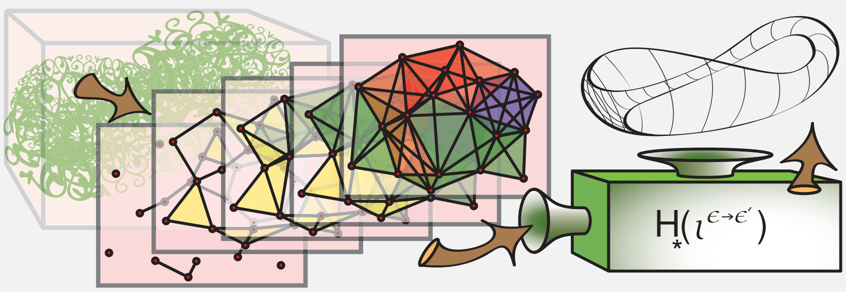

Figure 3 Persistent homology of a simplicial approximation finds hidden structures in large data sets.

What does the data look like?

Problem: High-Dimensional Data Analysis. Given a large, high-dimensional data set, how can one determine its shape and structure?

Tool: Persistent Homology. Though the subject of topology is often introduced in terms of doughnuts, coffee cups, knots, or other visual icons, the true strength of topology is the ease with which it analyses high-dimensional objects. The impact of this strength is perhaps best asserted in data-analysis, where the incoming rate of large, high-dimensional data sets currently far exceeds statisticians' abilities to analyse and describe the data sets.

Assume for the sake of argument that one is given a data set that consists of a *sampling* (perhaps, though not necessarily random) of a reasonable subset $X \subset \mathbf{E}^n$ of Euclidean space. Nature has trained the human brain to reconstructing shapes from planar projections, but this works only for certain (small!) values of n.

Knowing the homology of X is a good basis for asserting the global features of the 'true' model X of the data. Several basic statistical ideas – *e.g., clustering* – are readily seen to correspond to something homological, in this example $\dim H_0$. The natural question presents itself: how can one compute $H_*(X)$ from a discrete sampling of points $N \subset X$?

The work of Carlsson et al. employs the following strategy. Fix a parameter $\epsilon > 0$, and build a simplicial complex R_ϵ as follows: a k-simplex of R_ϵ is a collection of $k+1$ data points in N pairwise within distance ϵ. Fixing $X \subset \mathbf{E}^n$ a manifold, then for ϵ sufficiently small and N sufficiently dense, the complex R_ϵ has the same homotopy type (and thus homology) as X. However, one is given a fixed data set, and further refinement maybe be expensive or impossible. Thus, one is forced to vary ϵ. Which ϵ best captures the true topology of the underlying data set? For ϵ too small, R_ϵ is a discrete set; for ϵ too large, R_ϵ is a single simplex. In this context, the golden mean may not exist.

Algebraic-topology suggests a functional approach. One of the simplest and best insights of the Grothendieck programme is the notion that the topology of a given space is framed in the *mappings* to or from that space. With this perspective as guide, one considers the ordered sequence of spaces $\{R_\epsilon\}$ for $\epsilon > 0$, stitched together by *inclusion maps* $\iota^{\epsilon\to\epsilon'} : R_\epsilon \hookrightarrow R_{\epsilon'}$ for $\epsilon < \epsilon'$. The homology of the family of maps $\iota^{\epsilon\to\epsilon'}$ is the called the *persistent homology* of the data set: $\iota_*^{\epsilon\to\epsilon'}$ captures which homological features (holes in the data set) *persist* over the range of parameters $[\epsilon, \epsilon']$.

Carlsson et al. use the classification of *modules* over a polynomial ring (with field coefficients) to compute persistent homology and to correlate it with the birth and death of topological features in the data [7]. This allows a principled and automatic distillation of complex data sets into global features — a method that does not rely on projections or heuristics.

Specific successes of the method include the following.

- Persistent homology was used [2] to find significant features hidden in a large data set of pixellated *natural images* compressed onto a 7-dimensional sphere; most notable is a persistent *Klein bottle* in H_2, which in turn yields insights into the structure of the space of natural images.
- Recent work [3] uses persistent homology to find hidden structures in experimental data associated with the *V1 visual cortex* of certain primates.

Moral: "The shape of the data lies not in a single space, but in a diagram of spaces."

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Research

Flows in railway optimization

The Netherlands has an intricate network of railway tracks to connect its many closely situated cities. Recently, the railroad timetable has been completely updated. Unlike the yearly manual adjustments of the past, this update was done using mathematical techniques. For the Dutch public, this was a rare opportunity to come into contact with mathematics. The research that resulted in the new timetable won a team of CWI consisting, besides Lex Schrijver, of Gábor Maróti and Adri Steenbeek the prestiguous Franz Edelman Award for Achievement in Operations Research and the Management Sciences.

In December 2006, *Nederlandse Spoorwegen* (NS, Dutch Railways) introduced the *Spoorboekje 2007* (Timetable 2007). It included a new structure of the train timetable and new plans for train circulation and crew scheduling. Several new connections were made and train lengths were better matched to the number of passengers, while on the other hand train times were scheduled less tightly, a number of transfers were cancelled and some direct train connections were cut into parts in order to reduce the propagation of delays.

Mathematicians were involved in designing the algorithms that created the new schedules. Until 2007, new timetables were made by 'manually' adapting the timetable introduced in 1975. The increase in passenger numbers and the addition of extra infrastructure (new lines and forks, four tracks, and fly-overs) was met mainly by shifting train times and inserting new trains between existing trains. The need for a completely new timetable arose since hardly any trains could be added to the existing schedule where needed. The schedulers at NS felt they needed algorithms to better exploit the capacity of the railway network, and that is why NS asked mathematicians.

While the Netherlands is not large, it is one of the most densely populated countries of the world. The Dutch railway network is correspondingly rather dense, with many short trajectories, run by frequent trains, having several transfer connections on the way. Moreover, space to extend infrastructure is limited in the Netherlands.

These characteristics mean that railway optimization in the Netherlands is faced with rather specific problems compared to most other countries and no adequate off-the-shelf software was therefore available. The algorithms for timetabling and train circulation were made at the Center for Mathematics and Computer Science (CWI) in Amsterdam, for crew scheduling at the University of Padova, and for routing trains through stations at the Erasmus University in Rotterdam.

As might have been expected, the changes raised much public discussion. In an editorial commentary, the Dutch nationwide newspaper *NRC Handelsblad* characterized the new timetable as "the only form of higher mathematics that arouses furious emotions in the country". In a reaction, the CEO of NS claimed that the new timetable was the best possible given the existing infrastructure.

Journalists phoned to ask me if indeed mathematically no better schedule was possible. My answer amounted to the fact that optimality depends on the boundary conditions, that is, in this case, the input of the algorithm: which trains and connections do you want to have, how often, where do trains stop, etc. This is not a question of mathematics but a question of company policy, and not only passenger comfort plays a role here but also other objectives and constraints like crew scheduling, profit, punctuality and infrastructure. Given these constraints and these choices made by the company, the algorithm searches for an optimum timetable.

Economically, the new timetable appears to be successful. The yearly profit of NS is projected to increase by 70 million euros due to the new schedules. On trajectories with the largest timetable improvements, the number of passengers has increased by 10–15%. The effect of this on societal economics and welfare has been estimated tentatively at hundreds of millions of euros. Moreover, despite more trains, the punctuality has increased and the economical effect of this is of the order of tens of millions of euros.

As mentioned, *Timetable 2007* means not only the introduction of a new timetable but also of new systems for rolling stock circulation, to increase seat availability for passengers, for crew scheduling, to improve train personnel rosters, and for routing trains through stations. At CWI we made algorithms for timetabling and for rolling stock circulation. Although the timetabling problem (in particular finding the cyclic, hourly pattern) is also interesting from a network point of view, in this article we just focus on rolling stock circulation, also because of its historical interest.

Transportation and circulation problems belong to the classical problems in operations research and are motivated by application to railways. Flow techniques are basic to it and we first describe two results that are of historical interest. After that, we get back to railway circulation at NS.

Cargo transportation in the Soviet Union

The earliest attributions to the mathematical study of transportation problems are usually to around 1940. The transportation problem was formulated by Hitchcock [4] and a cycle criterion for optimality was considered around the same time independently by Kantorovich in the Soviet Union and Koopmans in the USA. For political reasons, they published their results only later ([5,6]). When Kantorovich found his method, publishing about economics in the Soviet Union was very risky, as it was a politicized issue.

Koopmans, a Dutch mathematical economist who fled to the USA at the beginning of the Second World War, found his methods when appointed at the Combined Shipping Adjustment Board, a British-American agency that routed merchant ships during the Second World War, as they had to sail in convoys under military protection. In 1975, Kantorovich and Koopmans received the Nobel Prize in Economics for their work on transportation.

Less known is that earlier, in 1930, A.N. Tolstoĭ [7] found methods similar to those of Kantorovich and Koopmans. His article called *Methods of finding the minimal total kilometrage in cargo-transportation planning in space* was published in a book on transportation planning issued by the National Commissariat of Transportation of the Soviet Union.

Tolstoĭ presented a number of approaches for the transportation problem, including the now well-known idea that an optimum solution does not have any negative-cost cycle in its residual graph. This residual graph is obtained from the network by adding, to any arc on which the flow is positive, an arc in the reverse direction, with cost equal to the negative of the cost of the forward arc. Here 'cost' can be true cost, length or anything similar.

Tolstoĭ seems to be the first to observe that this cycle condition is necessary for optimality. Moreover, he assumed, but did not explicitly state or prove, the fact that checking the cycle condition is also sufficient for optimality.

Tolstoĭ illuminated his approach with applications to the transportation of salt, cement and other cargo between sources and destinations along the railway network of the Soviet Union. In the paper, he explains the solution of a concrete cargo transportation problem along the Soviet railway network. It has 10 sources and 68 destinations, and 155 links between sources and destinations, and is therefore quite large-scale for that time.

Tolstoĭ 'verifies' the solution by considering a number of cycles in the network and he concludes that his solution is optimum:

"Thus, by use of successive applications of the method of differences, followed by a verification of the results by the circle dependency, we managed to compose the transportation plan which results in the minimum total kilometrage."

Checking Tolstoĭ's problem with modern linear programming tools shows that his solution is indeed optimum.

Max-flow min-cut

The Soviet rail system also aroused the interest of the Americans and again it inspired fundamental research in optimization.

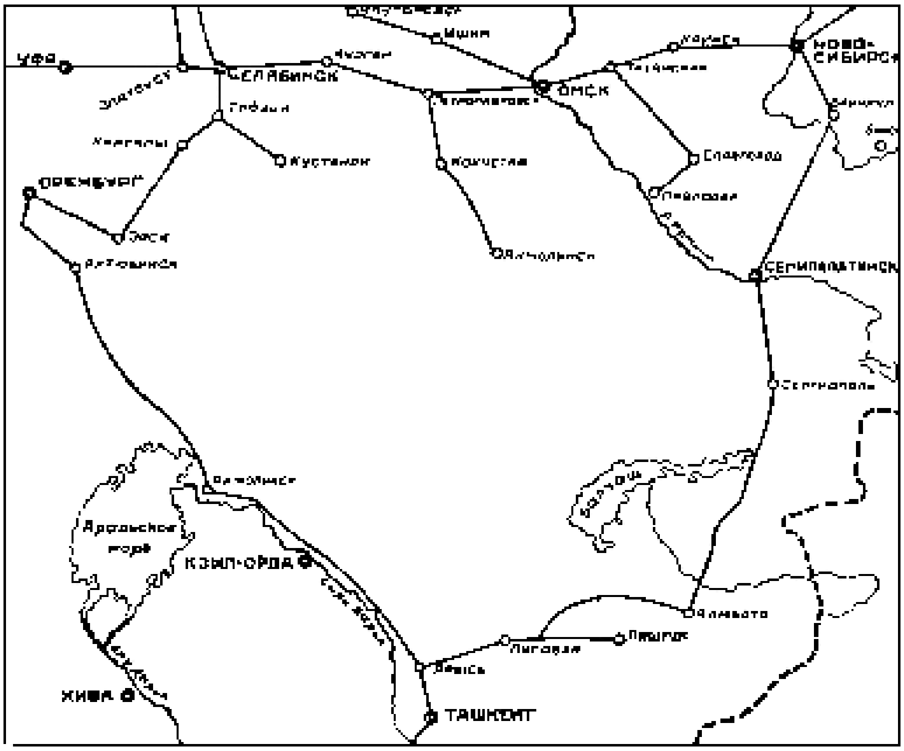

Figure 1 Figure from Tolstoĭ [7] to illustrate a negative cycle

In their basic paper *Maximal Flow through a Network* (published in 1954), Ford and Fulkerson [1] mention that the maximum flow problem was formulated to them by T.E. Harris as follows:

"Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other."

It inspired Ford and Fulkerson to their famous Max-Flow Min-Cut Theorem: *The maximum amount of flow that can be sent along a network from a set of sources to a set of destinations, subject to a given capacity upper bound, is equal to the minimum capacity of the cuts of the network that separate all sources from all destinations.*

In their 1962 book *Flows in Networks*, Ford and Fulkerson [2] give a more precise reference to the origin of the problem:

"It was posed to the authors in the spring of 1955 by T. E. Harris, who, in conjunction with General F. S. Ross (Ret.), had formulated a simplified model of railway traffic flow, and pinpointed this particular problem as the central one suggested by the model [11]."

Ford-Fulkerson's reference 11 is a secret report by Harris and Ross [3] entitled *Fundamentals of a Method for Evaluating Rail Net Capacities*, dated 24 October 1955 and written for the US Air Force. At our request, the Pentagon downgraded it to 'unclassified' on 21 May 1999.

In fact, the Harris-Ross report solves a relatively large-scale maximum flow problem coming from the railway network in the Western Soviet Union and Eastern Europe ('satellite countries'). And the interest of Harris and Ross was not to find a maximum flow but rather a minimum cut ('interdiction') of the Soviet railway system. (Recall that the report was written for the Air Force.) We quote:

"Air power is an effective means of interdicting an enemy's rail system, and such usage is a logical and important mission for this Arm. As in many military operations, however, the success of interdiction depends largely on how complete, accurate, and timely is the commander's information, particularly concerning the effect of his interdiction-program efforts on the enemy's capability to move men and supplies. This information should be available at the time the results are being achieved.

The present paper describes the fundamentals of a method intended to help the specialist who is engaged in estimating railway capabilities, so that he might more readily accomplish this purpose and thus assist the commander and his staff with greater efficiency than is possible at present."

The Harris-Ross report stresses that specialists remain needed to make up the model (which is always a good tactic to get new methods accepted):

"The ability to estimate with relative accuracy the capacity of single railway lines is largely an art. Specialists in this field have no authoritative text (insofar as the authors are informed) to guide their efforts, and very few individuals have either the experience or talent for this type of work. The authors assume that this job will continue to be done by the specialist."'

Whereas experts are needed to set up the model, to solve it is routine (when having the 'work sheets', which were added to the report).

The Harris-Ross report describes an application to the Soviet and East European railways. For the data it refers to several secret reports of the Central Intelligence Agency (CIA) on sections of the Soviet and East European railway networks. After the aggregation of railway divisions to vertices, the network has 44 vertices and 105 (undirected) edges.

The report applies flow techniques to obtain a maximum flow from sources in the Soviet Union to destinations in East European 'satellite' countries (Poland, Czechoslovakia, Austria and East Germany) but the main objective was to find the corresponding minimum cut separating the sources from the destinations. In the report, the minimum cut is indicated as 'The bottleneck' (see Figure 4).

While Tolstoĭ and Harris-Ross had the same railway network as object, their objectives were dual.

Figure 2 A 'koploper'

[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Amsterdam	V		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam	A		7.55	8.55	9.55	10.55	11.55	12.55	13.55	14.55	15.55	16.55	17.55	18.55	19.55	20.55	21.55	22.55	23.55
Rotterdam	V	7.00	8.02	9.02	10.02	11.02	12.02	13.02	14.02	15.02	16.02	17.02	18.01	19.02	20.02	21.02	22.02	[illegible]	
Roosendaal	A	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal	V	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			
Vlissingen	A	8.36	9.36	10.36	11.36	12.36	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21.36	22.36			

[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Vlissingen	V			[illegible]	[illegible]	[illegible]	8.58	9.58	10.58	11.58	12.58	13.58	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal	A			[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal	V		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam	A		[illegible]	7.25	[illegible]	9.32	10.32	11.32	12.32	13.32	14.32	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam	V	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Amsterdam	A	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Figure 3 Timetable Amsterdam-Vlissingen and Vlissingen-Amsterdam

Rolling stock circulation in the Netherlands

We will finally describe a more recent (and more peaceful) application of flow methods to railways, as used by Nederlandse Spoorwegen for *Timetable 2007*.

NS runs an hourly train service on its route Amsterdam-Rotterdam-Roosendaal-Vlissingen and vice versa, with the timetable shown above.

The trains have more stops but for our purposes only those given in the table are of interest since at the stations given train sections can be coupled or separated. For each of the stages of any scheduled train, NS has estimated the number of passengers, as given in the table on the next page (all data concerns weekdays and 2$^{\text{nd}}$ class seats).

The problem to be solved is:

What is the minimum amount of train stock necessary to perform this train service in such a way that at each stage there are enough seats?

In order to answer this question, one should know a number of further characteristics and constraints. In a first version of the problem considered, the train stock consisted of one type of two-way train units ('koplopers'), each consisting of three carriages. Each unit has 163 seats.

Each unit has at both ends an engineer's cabin and units can be coupled together up to a certain maximum length (often 15 carriages, meaning in this case 5 train units).

The train length can be changed, by coupling or decoupling units, at the terminal stations of the line, that is at Amsterdam and Vlissingen and *en route* at the intermediate stations Rotterdam and Roosendaal. Any train unit decoupled from a train arriving at place p at time t can be linked up to any other train departing from p at any time later than t (the Amsterdam-Vlissingen schedule is such that in practice this gives enough time to make the necessary switchings).

A last condition is that for each place $p \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen}, the number of train units staying overnight at p should be constant during the week (but may vary for different places). This requirement is made to facilitate surveying the stock and to equalize at any place the load of overnight cleaning and maintenance throughout the week. It is not required that the same train unit, after a night in Roosendaal, for example, should return to Roosendaal at the end of the day. Only the number of units is of importance.

Given these problem data and characteristics, one may ask for the minimum number of train units that should be available to perform the daily cycle of train rides required.

It is assumed that if there is sufficient stock for Monday till Friday then this should also be enough for the weekend services since at the weekend a few early trains are cancelled and on the remaining trains there is a smaller expected number of passengers. Moreover, it is not taken into consideration that stock can be exchanged during the day with other lines of the network. In practice this will happen but initially this possibility is ignored.

A network model

If only one type of railway stock is used, clas-

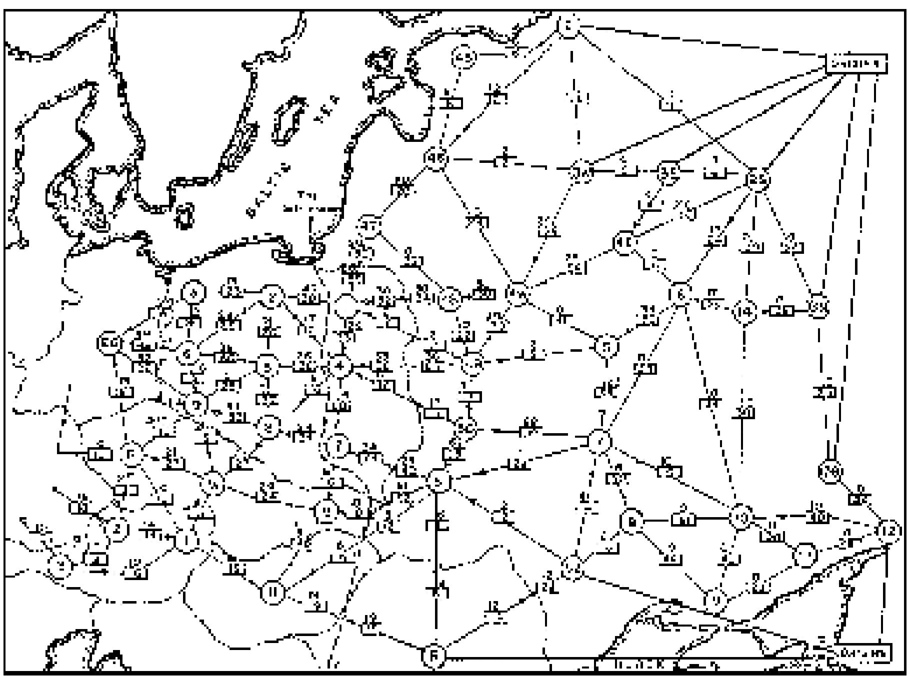

Figure 4 From Harris and Ross [3]: Schematic diagram of the railway network of the Western Soviet Union and East European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe and a cut of capacity 163,000 tons indicated as 'The bottleneck'

Train number	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Amsterdam-Rotterdam		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam-Roosendaal	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal-Vlissingen	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			

Train number	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Vlissingen-Roosendaal			[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal-Rotterdam		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam-Amsterdam	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Figure 5 Number of required seats

sical minimum-cost flow techniques (rooted in the work of Tolstoĭ) can be applied. To this end, a directed graph $G = (V, A)$ is constructed as follows. For each place $p \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen} and for each time t at which any train leaves or arrives at p, we make a vertex (p, t). So the vertices of G correspond to all 198 time entries in the timetable (Figure 3).

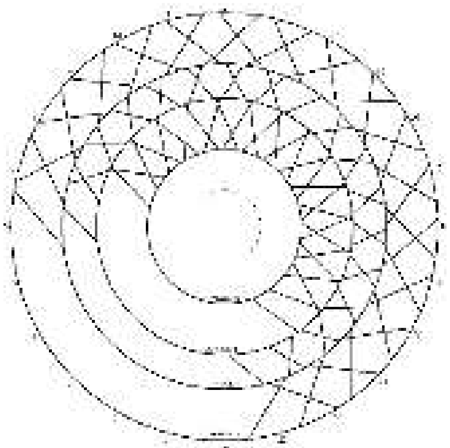

Figure 6 The graph G. All arcs are oriented clockwise.

For any stage of any train ride, leaving place p at time t and arriving at place q at time t', we make a directed arc from (p, t) to (q, t'). For instance, there is an arc from (Roosendaal, 7.43) to (Vlissingen, 8.38).

Moreover, for any place p and any two successive times t, t' at which any train leaves or arrives at p, we make an arc from (p, t) to (p, t'). Thus in our example there will be arcs for example from (Rotterdam, 8.01) to (Rotterdam, 8.32) and from (Rotterdam, 8.32) to (Rotterdam, 8.35).

Finally, for each place p there will be an arc from (p, t) to (p, t'), where t is the last time of the day at which any train leaves or arrives at p and where t' is the first time of the day at which any train leaves or arrives at p. So there is an arc from (Roosendaal, 23.54) to (Roosendaal, 5.29).

We can now describe any possible routing of train stock as a function $f : A \longrightarrow \mathbf{Z}_+$, where for any arc a, $f(a)$ denotes the following. If a corresponds to a ride stage, then $f(a)$ is the number of units deployed for that stage. So if a is the arc from (Roosendaal, 7.43) to (Vlissingen, 8.38), then $f(a) = 4$ means that 4 coupled train units will run this train stage. If a corresponds to an arc from (p, t) to (p, t'), then $f(a)$ is equal to the number of units present at place p in the time period t–t' (possibly overnight).

First of all, this function is a *circulation*. That is, at any vertex v of G one should have:

$$\sum_{a \in \delta^+(v)} f(a) = \sum_{a \in \delta^-(v)} f(a), \qquad (1)$$

the *flow conservation law*. Here $\delta^+(v)$ denotes the set of arcs of G that are entering vertex v and $\delta^-(v)$ denotes the set of arcs of G that are leaving v.

Moreover, in order to satisfy the demand and capacity constraints, f should satisfy the following condition for each arc a corresponding to a stage: $f(a) \le 5$ and $163 f(a) \ge d(a)$.

Here, $d(a)$ is the 'demand' for that stage, that is, the lower bound on the number of seats given in Figure 5 (and 163 is the number of seats of a Type 3 'koploper').

To find the total number of train units used by a given circulation f, one may add up the flow values on the four 'overnight arcs'. So if we wish to minimize the total number of units deployed, we minimize $\sum_{a \in A^\circ} f(a)$.

Here A° denotes the set of overnight arcs. So $|A^\circ| = 4$ in the Amsterdam-Vlissingen example.

It is easy to see that this fully models the problem. Hence determining the minimum number of train units amounts to solving a minimum-cost circulation problem, where the cost function is quite trivial: we have cost$(a) = 1$ if a is an overnight arc and cost$(a) = 0$ for all other arcs.

Having this model, standard minimum-cost flow algorithms give an optimum rolling stock circulation in a fraction of a second. It turns out that for the circulation described above, 22 train units are needed.

It is quite direct to modify and extend the model so as to contain several other problems. Instead of minimizing the number of train units one can minimize the amount of carriage-kilometres that should be made every day, or any linear combination of both quantities.

In addition, one can put an upper bound on the number of units that can be stored at any of the stations.

Instead of considering one line only, one can more generally consider *networks* of lines that share the same stock of railway material, including trains that are scheduled to be split or combined. Nederlandse Spoorwegen has hourly trains from Amsterdam, The Hague and Rotterdam to Enschede, Leeuwarden and Groningen that are combined into one train on a common trajectory. This network is called 'The North-East'.

If only one type of unit is employed for that network, each unit having the same capacity, the problem can be solved quickly even for large networks.

Mixing several types of train units

The problem becomes harder if there are several types of train units of different capacity that can be deployed for the train service and for that NS asked for the help of mathematicians. Clearly, if for each scheduled train we would prescribe the type of unit that should be deployed, the problem could be decomposed into separate problems of the type above. But if we do not make such a prescription and if different types can be coupled together to form a train of mixed composition, we should extend the model to a 'multi-commodity circulation' model.

Let us restrict ourselves to the case Amsterdam-Vlissingen again, where now we can deploy two types of two-way train units that can be coupled together. The two types are type 3, each unit of which consists of 3 carriages, and type 4, each unit of which consists of 4 carriages. Indeed, NS has 'koplopers' of both lengths. Types 3 and 4 have 163 and 218 seats, respectively.

Again, the demands of the train stages are given in Figure 5. The maximum number of carriages that can be in any train is again 15. This means that if a train consists of x units of type 3 and y units of type 4 then $3x+4y \leq 15$ should hold.

It is quite direct to extend the model above to the present case. Again we consider the directed graph $G = (V, A)$ as above. At each arc a, let $f(a)$ be the number of units of type 3 on the stage corresponding to a and let $g(a)$ similarly represent type 4. So both $f : A \longrightarrow \mathbf{Z}_+$ and $g : A \longrightarrow \mathbf{Z}_+$ are circulations, that is, each satisfies the flow circulation law (1). For each stage a, the capacity constraint is now $3f(a) + 4g(a) \leq 15$ and the demand constraint is $163f(a)+218g(a) \geq d(a)$, where again $d(a)$ denotes the number of required seats on stage a (Figure 5).

Let cost_3 and cost_4 represent the cost of purchasing one unit of type 3 and one unit of type 4, respectively. Although train units of type 4 are more expensive than those of type 3, they are cheaper per carriage; that is, $\text{cost}_3 < \text{cost}_4 < \frac{4}{3}\text{cost}_3$. This is due to the fact that engineer's cabins are relatively expensive.

Then we must minimize

$$\sum_{a \in A^\circ} \left(\text{cost}_3 f(a) + \text{cost}_4 g(a)\right).$$

However, there is an important further constraint that makes the problem much harder: it is required that at any of the four stations given (Amsterdam, Rotterdam, Roosendaal and Vlissingen) one may either couple units to or decouple units from a train but not both simultaneously. Moreover, one may couple fresh units only to the front of the train and decouple laid off units only from the rear. So if one must decouple a unit of type 3 from the train, it should be at the rear. Similar rules apply at the terminal stations.

This makes the problem more combinatorial, as the order of the different units in a train does matter. This does not fit directly in the circulation model described above and requires an extension.

To this end, one describes a train composition on a stage a by a vector $z(a)$ in $\mathbf{R}^{C_a}$. Here C_a is the set of compositions that are allowed for stage a. This takes the number of seats required for stage a into account and also the maximum train length. So for a certain stage a, one might have

$$\begin{aligned} C_a = \{&3333, 33333, 3334, 3343, 3433,\\ &4333, 3344, 3434, 4334, 3443, 4343,\\ &4433, 444, 4443, 4434, 4344, 3444\} \end{aligned}$$

Here 3433 (for instance) means that the train consists of four units, of types 3, 4, 3 and 3, respectively, seen from the front of the train. Then $z(a)_c$ is equal to 1 for precisely one $c \in C_a$ (namely, the composition c running stage a) and $z(a)_c = 0$ for all other $c \in C_a$. This can be described by integer linear inequalities, and the values of $f(a)$ and $g(a)$ are linear functions of $z(a)$. So we have an integer linear programming model.

A first attempt to include the coupling conditions is to add linear constraints on $z(a)$ and $z(a')$, where a and a' are consecutive stages of a train ride (like Amsterdam-Rotterdam and Rotterdam-Roosendaal of train 2127) so as to exclude transitions that are forbidden by the rule that units should be added only at the front of a train or removed only at the rear of the train and not both at the same stop.

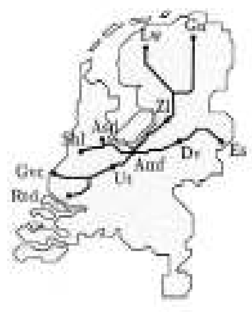

Figure 7 The *North-East*

This however turned out to be computationally infeasible. For those who understand integer linear programming, the linear relaxation of this problem is too loose to obtain good bounds in a branch-and-bound approach.

It took us a long time to realize that by adding even more variables, the problem can be solved relatively fast. For every *allowed* transition, say composition $c \in C_a$ on stage a is allowed to be followed by composition $c' \in C_{a'}$ in the subsequent stage a', we introduce a variable $w_{a,c,a',c'} \geq 0$ and require for every pair of two consecutive stages a and a':

$$z(a)_c = \sum_{c' \in C_{a'}} w_{a,c,a',c'} \text{ for all } c \in C_a \text{ and}$$

$$z(a')_{c'} = \sum_{c \in C_a} w_{a,c,a',c'} \text{ for all } c' \in C_{a'}.$$

This describes all constraints and, importantly, one does not need to require the new variables $w_{a,c,a',c'}$ to be integers. Therefore, the extra variables do not add to the complexity (the branching tree) but rather serve as oil to make the integer programming machinery work. With this, the problem turns out to be solvable with standard integer linear programming software in a few minutes. The number of extra variables is huge but this just helps you to find a solution within a few minutes.

It turns out that for the Amsterdam-Vlissingen problem, one needs 7 units of type 3 and 12 units of type 4. Comparing this solution with the solution for one type only, the possibility of having two types gives both a decrease in the total number of train units and in the total number of carriages needed. The method can be extended to include combinations and splits of trains at intermediate stations and thus also applies to the larger network 'The North-East', where similar savings have been obtained and more passengers can find a seat.

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History

Exploits in the Dutch history of mathematical sciences

Why did no one tell me that I was so close to Huygens, that historical automatons were within half an hours ride, that I could have seen more of Mondrian than any mathematician could digest! The following brief survey is meant to help you avoid such frustration after coming home from 5ECM.

A trip to the historical city of Amsterdam is not complete without visiting one of the famous Amsterdam museums appearing in the most common tourist guides (like the Rijksmuseum, the Van Gogh museum and the Stedelijk museum) but the Amsterdam Archive would like to present to you some additional Dutch exploits in the history of mathematical science and the related arts. We assume that the reader will be able to find windmills, dykes and Deltaworks without assistance.

Automatons

Those who are fascinated by automatons, and even those who can't suppress a smile at the sight and sound of a music machine, must find their way to Utrecht, located near the centre of the Netherlands, less than half an hour by train from Amsterdam. In the historical town centre lies the museum *Van Speelklok tot Pierement*. Located in a former church the museum houses a living collection of mechanical music: Carillon clocks, music boxes, Pianolas, belly organs and orchestrions, as well as full-sized street, fairground and dance hall organs. A tour of this museum tells the story of automated musical instruments through the ages. Of course, live demonstrations of the instruments are all part of the visit.

New in the museum is an automaton of Vincent van Gogh and, thanks to the fact that the museum has the world's number one restoration shop, three antique musical clocks from China on loan from the imperial collection of the Palace Museum in Beijing.

A larger detour, but certainly worth it, will take the lover of automatons to Franeker, further north in Friesland, to see one of the greatest orreries in history: Eise Eisenga's planetarium — built in 1774–1781 and still in operation.

History of the mathematical sciences

From the same age dates the Netherlands' oldest museum *Teylers Museum* (1784) in Haarlem. The railway track leading west from Amsterdam to Haarlem is a memorable track, since it was the first track to be constructed and used in the Netherlands. The city's old market square is dominated by St. Bavo's cathedral with its huge and nicely restored cylinder organ, which once attracted Mozart to play on it. Teylers museum, around the corner from the old market, is a museum that has been preserved in its original 18th century form. In ancient cabinets it presents drawings, prints and paintings, fossils and minerals, scientific instruments and coins and medals. It has a fine collection of drawings by Michelangelo, Rembrandt and others. Teylers' major attraction is the number of instruments showing the state of the mathematical sciences in the late 18th century, for example from the world's largest electrostatic generator — imagine those sparks flying.

Fifteen minutes from Haarlem, and ten minutes by train from Schiphol, is Leiden with its famous old university (1575). A memorial stone of mathematician Ludolph van Ceulen in *the Pieterskerk* shows why π was once called *the Ludophian number*. From 1600 onwards Leiden University had a school of *Duytsche Mathematique* (practical mathematics taught in Dutch) and Van Ceulen was among its first professors. An interest in Christiaan Huygens is a good reason to come to Leiden. The Scaliger Institute of the university library holds for the connoisseur exquisite treasures like the manuscripts of Huygens.

Much more accessible, in the *Museum Boerhaave*, are Huygens' clocks, lenses, orreries and telescopes. The Boerhaave Museum's history began in 1907, when an historical exhibition of the natural sciences and medicine was held in the academy building of Leiden University. The success of the exhibition provided the impetus for the founding of the Netherlands Historical Science Museum, the forerunner of today's national museum for the history of science and medicine, better known as the Boerhaave Museum. In terms of the history of science and medicine, the collections in the Boerhaave Museum are among the most important in the world. Early modern science is on display; do not miss the world's oldest herbarium. From the golden age of the 17th century not only Huygens' material is presented but also Willem Blaeu's giant quadrant, and microscopes by Antoni van Leeuwenhoek. The 18th century is splendidly represented by the cabinets — science demonstration laboratories — of Professors Gravesande and Van Musschenbroek. The huge quantity of 19th century objects includes Dr Zander's physiotherapeutic devices and papier-mâché anatomical models. The early 20th century, the scientific era known as the second golden age, is another highlight in the museum showing the legacy of Dutch Nobel Prize winners Van 't Hoff, Lorentz, Zeeman, Van der Waals, Kamerlingh Onnes and Willem Einthoven.

To further indulge in a Huygens museum one will want to visit the Huygens family estate *Hofwijck* in Voorburg, near The Hague, designed in a harmony of 'mathematical' relations by Christiaan Huygens.

Mathematics in art

As can be judged from another contribution in this issue, it is no longer fashionable for mathematicians to confess a liking for Escher. One does not have to travel to the Netherlands to find Escheriania abound in antiquarian bookshops. For those who cannot control their, now covert, weak spot for the prints of Maurits Cornelis Escher (1898–1972), there is a permanent museum in The Hague, and after 5ECM there is the 'Bridges' conference in Leeuwarden with an occasional exhibition at Escher's place of birth.

By contrast mathematicians are on their way to a reappraisal of Mondrian. Visit the Gemeente Museum Den Haag for the richest collection of his work, or the Mondriaanhuis in Amersfoort to see more of Mondrian's life and constructive and concrete art.

Rinus Roelofs (see the cover of this issue), the new star on the bridge between mathematics and art, will be present in person at the Leeuwarden Bridges Conference. Otherwise, in the absence of a real world Roelofs museum, rush home and consult his wonderful web site.

Websites

- Van Speelklok tot Pierement http://www.museumspeelklok.nl
- Teylers museum http://www.teylersmuseum.nl
- Museum Boerhaave http://www.museumboerhaave.nl
- Pieterskerk Leiden http://www.pieterskerk.com
- Hofwijck Voorburg http://www.hofwijck.nl
- Escher in het Paleis http://www.escherinhetpaleis.nl
- Rinus Roelofs http://www.rinusroelofs.nl

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Windmills near Zaandam

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Interview N.G. de Bruijn and Hendrik Lenstra

Escher for the mathematician

In 1954, the International Congress of Mathematicians was held in Amsterdam. For the occasion, a special Escher exhibition was organized in the Stedelijk Museum. Professor N. G. de Bruijn (1918) was one of the people involved. About half a century later, Professor Hendrik Lenstra (1949) stopped at Escher's Print Gallery. Why is there a hole in the middle? Can it be filled, or is there a reason why this could not be done? He did get an answer to these questions, and since then he and his colleagues have travelled around the world giving lectures about it. Jeanine Daems visited De Bruijn and Lenstra and asked them about these episodes in their lives and about their ideas about Escher's work.

N.G. de Bruijn and Hendrik Lenstra both have been involved with the work of M. C. Escher. What is their opinion of his work?

Interview with N.G. De Bruijn

In what way did you come to know Escher's work? I knew Escher's work quite some time before 1954. He was not really well-known at the time. Still, every once in a while an article on his work would appear. Moreover, I happened to get a real Escher print in 1946.

In February 1945 (it was the last year of World War II) a Temporary Academy was set up in Eindhoven. The southern part of the Netherlands had been liberated, while the rest was still under German occupation. In the south, quite a few young people wanted to go to university but there was no place to study. In Eindhoven, with many scientists around, it was easy to find staff. I happened to work in Eindhoven with Philips from 1944 to 1946, which had saved me from forced labour in Germany, so I could give several courses in that academy.

At the dissolution of the academy in December 1945, Escher was asked to make this commemorative picture and it was presented to all the lecturers (but not to the students). I think about 80 copies were made.

There is a river in it that also appears in a famous picture of Escher's, with white on one side of it and black on the other, with reflection; you can see something of that in this one. But this picture is mainly about the owl sitting on the ruins and the broken chains, while there are still fires burning on the other side of the river.

When in 1954 the International Congress of Mathematicians took place in Amsterdam, an exhibition of Escher's work was attached. The idea came up in a discussion with my friend Seidel who was to be the leader of the entertainment activities around the congress. I was the secretary of the Program Committee, and in that capacity I was in close touch with the organizers of the congress. They appreciated the idea at once. Then I approached Escher; visiting him at his home (in Baarn) and I asked him how he felt about it. He liked it very much. Then I went to the managing director of the Rijksmuseum.

The Rijksmuseum? Yes, I went to the Rijksmuseum since several congress manifestations were to be held there. It was the kind of centre where everyone would go because it was world-famous. But the managing director replied: "No, we will not do this; it is modern art! You should be at the Stedelijk Museum for that". Well, at the time I already held the opinion that Escher's work was not art at all, not beautiful at all but very clever and very interesting. Escher was not a mathematician; and no artist. Escher was Escher. Nevertheless the Stedelijk Museum rather liked the plan.

The exhibition had to be opened, of course. I thought Freudenthal should do that because I considered him to be the leading mathematician in the Netherlands. But the congress committee — Koksma and others — said to me: "No, it was your idea, you should do it yourself". So I thought, okay, let's do it then. There I really overplayed my hand, having no idea what that meant — opening an exhibition. I expected that I would have to stand

The catalogue of the exhibition in 1954

on a crate in the exhibition room and say: “I now declare the exhibition to be opened.”

How different it turned out. I was welcomed by the museum director and some other officials, such as the cultural deputy of the province of Noord-Holland. We had a cup of coffee and at some point they said: “Let us get started”. I had no idea what was supposed to happen. First, one of the officials was to give a brief introduction and next I was to give the main speech. But I did not have a speech! We entered this *hall*! It was filled with people; there must have been over 150. But I had luck: this official gave a very long speech first - it took at least fifteen minutes. After that, I did not speak for more than five minutes, using the few notes I had made meanwhile. The effect was that there were many people who thought: "That first speaker took much too long but the second one did a good job!"

Some of my remarks found their way into the printed catalogue and Escher would quote them with pride later. They were not about the mathematical content of Escher’s work but about the playfulness of it, the same kind of feeling we know from mathematics.

Mathematically, there was just one thing I had to report to Escher before the exhibition. I showed him pictures of the Klein model of hyperbolic space: the disk. He did not understand it at all, however, and since there were no photocopying machines yet my hint was lost. After the exhibition Coxeter approached Escher on this, and that made Escher start. Very nice, indeed. Previously, Escher had worked only with tessellations in connection with the 17 wallpaper groups. To Escher, who did not know any group theory, the 17 groups were just systems to make wallpaper. Those systems he knew: he had found most of them himself. Coxeter kept in touch with Escher in the years after the exhibition.

By the way, recent historical research has revealed even earlier contacts between Escher and George Pólya, who had written an article about the 17 symmetry groups in 1924 [Schattschneider, 2004].

Another interesting person whom Escher met at the congress was young Roger Penrose. His interest at that time was not tessellations but impossible figures, like stairs with water running up. As far as tessellations are concerned, I think it is a pity that Escher did not live long enough to see Penrose’s famous nonperiodic tilings with imperfect fivefold symmetry in the early 1970s. Escher might have made nice tessellations out of those.

How did people react at the exhibition? Many people went there. It turned out to be one of the best attractions of the congress. In the Netherlands it did not have much influence, but the exhibition sparked off his fame in the international scientific world.

What do you think is the most beautiful in Escher’s work? The most beautiful... well, you know, if it comes to beauty I am not really such an admirer of Escher.

You are not? No, you see, I understand how it was created, but Escher’s artistic interpretation was somewhat wooden, out of sheer necessity. His human figures looked like wooden puppets. That is why regular artists did not want to call it art; what he did was not something they could recognize. But we like it, don’t we?

Escher worked in black and white mainly. Had he used more colours, he might have produced more interesting symmetry issues. But that was difficult with the techniques he used. He was a great craftsman, of course, in a profession where a lot of patience and a lot of inspiration are required. Well, for mathematicians, Escher’s work is definitely entertaining, but beautiful... What is beauty? I really do not know.

Interview with Hendrik Lenstra

Given De Bruijn’s appreciation, what do you hold of Escher as an artist? Well, looking at Escher’s work, there is no denying that he could not draw! He could not draw people, for instance. It is a bit like those Willink paintings: if in a painting by Willink a man walks down the street then it is not a walking person; it is a dummy in the shape of a walking person. Escher could not draw, and he admitted that himself as I read in his biography. But it is not done to criticize Escher. In my opinion it is an indisputable fact that Escher could not draw, and it shows in many of his prints.

How did you get the idea to look at the Print Gallery *from a mathematical point of view?* Well, being a mathematician, I see everything from a mathematical point of view! I got acquainted with Escher’s work in highschool. In the early 1960s, Bruno Ernst started the highschool mathematics magazine *Pythagoras*. There he wrote articles about Escher’s work, in a nice way, at the right level, but the most fascinating were the pictures. The *Print Gallery* was one of them. As a teenager I bought books on Escher but, to be honest, when I became a mathematics student, I considered it somewhat inferior: it was not ‘real mathematics’.

With age one looks at things differently. As a mathematician you see certain things and you realise you do not understand them. I remember being on an airplane, reading about an Escher exhibition in one of these airline glossies. Here was the *Print Gallery* again and this time I looked at it from a completely different point of view. Clearly there was mathematics involved but I did not know *which* mathematics. In a way, the problem was my inability to formulate the problem. It is quite obvious what happens in the picture, and what Escher wanted to express. In

From the Foreword of the Catalogue of 1954

In view of the fact that Mr. Escher’s work may be said to be a point of contact between art and mathematics, the Organizing Committee of the International Congress of Mathematicians 1954 (2-9 September) at Amsterdam took the initiative in inaugurating this exhibition.

Probably mathematicians will not only be interested in the geometrical motifs; the same playfulness, which constantly appears in mathematics in general and which to a great many mathematicians is the peculiar charm of their subject, will be a more important element.

It will give the members of the congress a great deal of pleasure to recognize their own ideas, interpreted by quite different means than those they are accustomed to using. *N.G. de Bruijn*

Image by Henry Segermanand Paul-Olivier Dehaye after an idea of Hendrik Lenstra

Hendrik Lenstra in front of the Memorial Church in Stanford

Escher literature it is called *circular expansion*. I tried an indirect approach: what if someone had this artistic idea and came to me during office hours — for the sake of argument, let us assume I had office hours — and asked me: "I want to express circular expansion. How should I do that? What does mathematics have to offer in this?". This is a way to ask the question, though not a very precise way. It is a question of the kind that applied mathematicians are more likely to be confronted with than pure mathematicians. Pure mathematicians have certain questions but these are mathematical questions already. The applied mathematician, by contrast, is confronted with the phenomena of the outside world, and the first step is to turn these into mathematics. To me that was the biggest problem: how to turn it into mathematics — how to express it in a mathematical way. Another thing that I wondered about, just to have a question to think about, was the hole in the middle of the *Print Gallery*. Some people say it is ugly, but that is not very relevant. My question was: is it necessary? There is mathematics involved and once the relevant mathematics is understood, obviously, one should be able to answer the question of whether the hole can be filled, or if there is a limit beyond which the filling would become a farce. But what does 'a farce' mean? I had not put that into words yet.

The drafts Escher made when working on

the *Print Gallery* had been published. Grids on these drafts show clearly what happens in the middle. I can imagine that Escher could see that, too. But I kept wondering about the mathematics; which functions were involved, for instance. Escher made four drafts, and every draft turned up four times smaller in the next draft. Evidently, there was a *Droste effect*. Eventually I arrived at a good question. In the drafts, I could see the picture was preserved under scaling by a factor $4^4 = 256$. I knew there had to be a repetition in the *Print Gallery* as well. This could be observed from the grid underlying the picture, but measurement of distances on the grid showed that the repetition in the grid was very different from that in the drafts. There was a rotation in the picture and this factor 256 was nowhere to be seen! There was a factor of about 20 and a rotation of 160 degrees. That was odd. So my question was: what are these numbers? And that was a precise question. I had an answer about five minutes later because it had become clear from the book by Bruno Ernst that the key concept was conformality. The transformation had to be conformal, which means that angles are preserved. Remember I asked the question: what would 'a farce' mean? Well, that means that the distortion is such that details are no longer recognizable. Bruno Ernst mentioned in his book that Escher was forced to bend the straight lines a bit because in that way the little squares in the grid would remain more like squares.

This made me realize that the drafts that I was looking at actually presented a quite well-known alternative way to describe elliptic curves over the complex numbers. The *Print Gallery* itself was drawn with a different period, and it was also an elliptic curve. And if the transformation between them is conformal then the elliptic curves have to be isomorphic. Well, I know everything about isomorphisms between elliptic curves, so I could answer my questions immediately. That was in January 2000. By now, the research question had turned into a nice topic to discuss over lunch.

I gave the first lecture about it in March 2000. Robbert Dijkgraaf was present and gave a new impulse to the work by asking: "Why don't you programme that?". Well, I did not because I am not a computer programmer.

The main people involved in the project were myself (I did not do much, though I had some money to spend on it) and Bart de Smit. And we needed someone for the programming work — that was Joost Batenburg, who was a student here in Leiden at the time.

As a project it really gained momentum when Sara Robinson, a science writer, proposed to write an article on it for the *New York Times*. We liked that, and looking ahead we realized that the article would generate a lot of questions and emails. We needed a place we could direct people to, so we made a website (escherdroste.math.leidenuniv.nl), which was launched the day the article appeared. Without Sara we would not have worked so hard on this.

Also, we had two artists working with us. The first was Hans Richter. We had to fill the hole in the middle. Because of the repetition, a considerable part of the hole can be filled using Escher's picture itself. But not all of it; a small part remained empty. In the straightened picture, this corresponded to an empty spiral. It was clear that it was possible to fill it; one would have to finish Escher's straight picture: continue some lines and so on. Richter did that. He was very enthusiastic about it; he understood the mathematics and he even found an inconsistency in Escher's picture: certain lines that should be parallel are not. Actually, Richter had to draw a bit himself: in our version of the *Print Gallery* there was a bit more space in the actual gallery and he drew his favourite Escher picture there: the Möbius strip with the ants. That was funny because this Möbius strip was made in 1963, whereas the *Print Gallery* is from 1956. Fantastic! Anachronisms. The impossible. That fits with Escher! The other artist was Jacqueline Hofstra: she 'coloured' the picture with shades of gray. She was very surprised at my enthusiasm about the anachronism. She had not expected mathematicians to like impossible things. People have strange ideas about mathematicians.

The project did, in fact, attract a good deal of publicity and we have given many lectures all over the world.

What do you think is interesting in Escher's work? What I find interesting are the things I do not understand. But I have to add that 'to understand' might not mean the same to different people. When I ask myself whether I understand something then I do not always find it easy to test whether I do. That is why I thought of questions like: when an artist comes into my office and asks this question, do I know what to tell him? And now I feel that I do. You have to check what mathematics this artist knows and on what level I can explain things to him. But I now know what to say. When one really understands a mathematical theory, one can explain it at any level.

But there are other pictures by Escher that I do not understand in this way. Sometimes it is just that I do not know the field, as is the case with hyperbolic geometry. I can imagine someone writing a book about that: 'Escher for mathematicians'. That could be a lovely book. It should not be aimed at high-school students but at mathematicians, and it should explain things in the way you would like to understand it yourself. I think a thorough examination of all of Escher's pictures would cover many areas of mathematics.

The ideas behind the *Print Gallery* could be applied by any artist. I am not really into doing that myself. It can be fun on occasion: we transformed Rembrandt's *Night Watch* and put that on the wall of our lunch room. It is a bit of a puzzle to get an artistically satisfactory result, but it is a kind of automatism. And we do not do automatisms in mathematics! When we understand something, it is no longer interesting.

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Education

Where have all the students gone?

Foreign colleagues are usually flabbergasted when they hear that the number of mathematics students in the Netherlands has declined by about 70% between 1975 and 2005, in a world where the paramount importance of mathematics has become increasingly evident. Indeed, currently Holland as a whole sports about as many mathematics students as a single typical large university in a neighbouring country. How can this be? What is being done to turn the tide? The author, Klaas Landsman, is professor of Mathematical Physics at the Radboud University Nijmegen. He was the principal applicant of the GQT-cluster and was a member of the Mathematics Soundboard **of the ministry of Education, Culture and Science. He is currently one of the authors of the** Masterplan Toekomst Wiskunde**, in which the long-term future of Dutch mathematics is being laid out.**

Holland is one of the wealthiest countries in the world. Its technological infrastructure is impressive and Dutch civil engineers enjoy a worldwide reputation. At the beginning of the 20th century, Dutch physicists won one Nobel Prize after another. Multinationals like Philips, Shell and AKZO-Nobel rely heavily on research and development. Hence one would expect to find an exemplary educational system, culminating in science students formidable in both quantity and quality.

Diagnosis

Whoops! In actual fact, at the time of writing an official 'parliamentary enquiry' has just come to an end. Its goal was to find out how standards of learning could possibly have deteriorated so much over the past 30 years that 40% of our first-year *university* students are now unable to spell even elementary verbs correctly and more than half of the science and economics students fail elementary algebra tests. The situation at teacher training colleges is equally desperate. Serious Dutch newspapers cover this theme on a daily basis, sometimes even featuring some new negative educational statistics of the above kind as breaking news on their front page. Our population is getting increasingly nervous, politicians following in the wake — at last!

For us, as mathematicians, the main effect of this general demise of learning has been a dramatic decrease in the number of mathematics students: in 1975 about 700 students — already a less than impressive number — entered an undergraduate mathematics degree program at some general or technical university but in 2005 the number had dropped below 200. In addition, even though topics like differentiation and integration of functions of a single real variable have remained part of the mathematics curriculum at school, genuine insight into these and other mathematical operations among schoolchildren is rare. At school, most computations are nowadays performed on an electronic pocket calculator and algebraic formulae are simply copied from a 'formula card' without any understanding of their derivation. Only a few prodigies are able to produce a correct deductive argument, let alone a proof of a theorem.

Anamnesis

Part of the decline in mathematics students may be accounted for by the rise of computer science since 1975, which has certainly attracted students who otherwise would have chosen mathematics. Also, the idea that the biosciences have replaced the hard sciences at the frontier of human knowledge has undoubtedly played a role. But these arguments are not peculiar to the Netherlands, whereas the situation sketched above surely is. Thus we have to look for reasons unique to Holland, if only to find out how to reverse the downward trend.

Two major factors appear to have played a role. First, over the past thirty years the teaching profession as a whole has been systematically undermined by a combination of policies. These include:

- Salary cuts;
- Loss of power and influence to school managers;
- Educational reforms.

The salary cuts for teachers, which went beyond those for civil servants, were among the financial measures taken by Prime Minister Lubbers and his various governments from 1982 onwards in response to gross overspending by his predecessors Den Uyl (1973–1977) and Van Agt (1977–1982). The trade unions had their way as well, in that starters had to carry the main burden. The management layer at schools used to consist of the teachers themselves but began to form a separate caste in the wake of government-demanded mergers, which led to schools with thousands of pupils. The reforms — going under the name of 'new learning' — aimed at replacing teachers with 'coaches' who no longer teach but admire their pupils whilst they find out the truth — necessarily subjective — themselves by, for example, surfing the Internet.

Consequently, starting a career in teaching became an unattractive option in many ways. Since especially those with a university degree had other opportunities, the proportion of university-educated teachers has dropped substantially compared with teachers who obtained their qualification from a teacher-training college, or, indeed, have no teaching qualification at all. The latter phenomenon is especially common in mathematics, in which there is such a shortage of qualified teachers that schools are desperately trying to fill their vacancies with whoever is simply willing to teach mathematics, be it an economics teacher or a former driving instructor.

The second factor is slightly controversial, although academic mathematicians appear to be united in bringing it up even as the main culprit. In the mid-80s, the Dutch mathematics curriculum in secondary education was drastically reformed in order to make mathematics 'realistic'. In fact, what this has come to mean in the Netherlands is that children learn a bag of tricks, which they are supposed to apply to problems typically posed to them in the form of stories. Since genuine applications of mathematics to science or society would require some previous stage of abstraction, these stories are actually rarely realistic at all, typically involving completely artificial if not infantile settings. What little theory and abstraction has remained in textbooks is frequently remote from serious mathematics and is sometimes even plainly erroneous.

The introduction of 'realistic' mathematics was partly a response to the 'New Math' ideology of the 60s, which in its extreme implementations based highschool mathematics on set theory and even in softer versions made the subject far too abstract and inaccessible for the average adolescent. However, it seems equally wrong to remove practically all abstraction, as in the 'realistic' ideology: with the loss of the very essence and power of mathematics, namely the *interaction* between abstraction and application, the baby has been thrown out with the bath water.

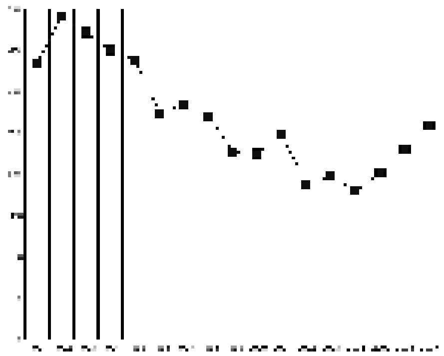
Those responsible for the 'realistic' mathematics program would typically say that mathematics has become more palpable this way, so that — as allegedly shown by PISA (Programme for International Student Assessment) tests — the average level of mathematical understanding among the Dutch school population has risen since it was introduced. The interested reader is referred to the critical literature on PISA for a rebuttal of such claims [1]. For me, it suffices that the numerous schoolchildren I have been in close contact with over the past few years during promotional activities of the kind described below *themselves* complain that they understand neither what mathematics is nor why it is important to science or society. Similarly, in an unprecedented petition called *Lieve Maria* (*Dear Mary*), offered to our previous Minister of Education, Culture and Science Mrs Maria van der Hoeven in January 2006, the 10,000 signatories *themselves* complained that their mathematical training at secondary school had been insufficient.

Treatment

Initially, with a few exceptions, the response from the academic community to the steady drop in enthusiasm for mathematics among teenagers and the concordant decline in students was lukewarm, not to say indifferent. One professor is even on record as saying that he welcomed this decline, as it gave him fewer exams to mark. Fortunately, this introvert attitude — which reminds one of the avoidable assassination of Archimedes — has decisively changed over the last five years. Indeed, academic mathematicians began to feel the impact of low student numbers through dras-

Photographer: Dick van Aalst

From left ro right: Mirte Dekkers, Dieuwertje Ewalts, two high school students, Ruben van den Brink, Klaas Landsman, Jozef Steenbrink


Number of first-year mathematics students in the Netherlands (1991–2007)

tic cuts in their own numbers. At Nijmegen these went so far that the Dean of the Faculty of Science even decided to close down the entire mathematics department. This decision was revoked after nationwide and international protests but the community had been warned and began to take action at last!

In view of such immediate threats, one of the earliest initiatives was not directed at schoolchildren but rather at the universities themselves. Prompted by mathematicians Marinus Kaashoek and Henk van der Vorst in 2002, the Netherlands Organization for Scientific Research (NWO), in conjunction with both the Ministry of Education, Culture and Science and the Ministry of Economic Affairs, made millions of euros available for mathematics research from 2005 onwards, provided this research was to be carried out collaboratively in so-called *clusters*. At the moment, three such clusters are active:

- DIAMANT, standing for *Discrete, Interactive & Algorithmic Mathematics, Algebra & Number Theory*, a collaboration between Eindhoven Technical University, Leiden University, Radboud University Nijmegen and the National Research Center for Mathematics and Computer Science in Amsterdam;
- GQT, i.e. the *Fellowship of Geometry and Quantum Theory* in which the University of Amsterdam, the Radboud University Nijmegen and Utrecht University take part;
- NDNS+, for *Nonlinear Dynamics of Natural Systems*, involving the University of Groningen, Leiden University, the Vrije Universiteit Amsterdam and the Center for Mathematics and Computer Science.

Apart from a renewed élan of Dutch mathematical research as a whole, the main effect of these clusters so far has been that further budget cuts appear to have been avoided, at least in the areas involved (typically, faculty positions gained by the clusters were lost elsewhere in a given mathematics department). On the other hand, areas like stochastics and logic, which have not been organized into a cluster, remain fragmented with huge imbalances between universities.

An enterprise that *will* directly affect secondary school mathematics is the preparation of a wholesale reform of the curriculum due in 2012 (this is being done for all the sciences). In 2004, the Ministry of Education, Culture and Science charged a *Committee for the Future of Mathematics Teaching* chaired by Dirk Siersma with the difficult task of overcoming the ideological conflict between the academic community and the educational establishment and drawing up a new exam program. Dutch Parliament subsequently called for the installation of a second committee, the *Mathematics Soundboard* chaired by Jan van de Craats, whose job it was to assess the relevance of the new programs to higher education and advise the ministry to amend these programs if necessary. After all, it is higher education that is suffering most from the ‘realistic’ mathematics curriculum. The latter committee also included three students, two of whom were involved in the *Lieve Maria* petition. At the end of the day, to the satisfaction of most this process has resulted in a balanced curriculum in which abstraction and application both play a central role.

Of course, the universities cannot wait until this program takes effect. Meanwhile, we literally go out of our way to show teenagers how mathematics can *really* be applied to science and society, precisely because of the availability of some abstract theory also displaying the intrinsic beauty of the subject. As an additional bonus, in showing that mathematics is ubiquitous, one simultaneously makes it respectable in the eyes of the general public (instead of a source of misunderstanding if not derision). This is particularly effective if mathematics is combined with a certain measure of success; think of former geometer Jim Simons’ hedge fund *Renaissance Technologies*, whose secret mathematical trading strategy made him a billionaire. Alternatively, consider the mathematics behind Google, MP3 players or mobile phones. As pointed out by visionary PhD student Ruben van den Brink, immersing mathematics into society and being proud of it will actually have a positive effect on student numbers as well, for the reason that schoolchildren contemplating a mathematics degree are now going to be admired by their friends, rather than being evaded as nerds. Given the undeniable fact that mathematics is very hard indeed, such admiration may provide the decisive push in actually going for such a degree.

This philosophy is brought into practice through activities like master classes and web classes for upper level schoolchildren (every mathematics department in Holland now organizes at least one of these), help in writing mathematics essays, mathematical summer camps and a yearly mathematics tournament at Nijmegen for 100 teams of 5 teenagers each, with a trip to New York for the two winning teams. Such activities are accompanied by campaigns organized by both industry (combined in the *Jetnet* platform) and the Government, who organize numerous activities to point out the importance of the sciences as a whole. In the public sphere, the *Platform Bèta Techniek* is a particularly efficient vehicle, through which millions of euros are spent each year on a chain of programs starting at primary school and ending at the labour market.

Finally, visibility of Dutch mathematicians in the media has increased markedly. Here Robbert Dijkgraaf leads by example as a TV personality and regular newspaper columnist, besides his normal activities in string theory and mathematical physics, supplemented by his recent appointment as President of the Royal Netherlands Academy of Science (KNAW). Last summer, he and another renowned Dutch mathematician, Hendrik Lenstra, even became pop stars for one day through their appearance at the Lowlands Festival, where Dijkgraaf gave a talk about Einstein and Lenstra explained the mathematics behind Escher’s drawings.

Has any of this really helped? Yes, it has! In 2005 less than 200 students enrolled in an undergraduate mathematics degree programme; in 2007 the number was just above 300. This is still a far cry from the 700 it once was but with applications for 2008 once again on the rise, the worst crisis appears to have been overcome and the future for Dutch mathematics looks bright!

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1 The unreliability of the PISA tests was one of the conclusions of the parliamentary enquiry mentioned at the beginning of this article. See also www.beteronderwijsnederland.nl/?q=node/1340. Although this web site is in Dutch, it links to a large number of pertinent documents in English. Another relevant website is www.math.nyu.edu/~braams/links.

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Society

Confessions of an industrial mathematician

Mathematical Study Groups with Industry provide since long a unique opportunity for mathematicians, from young to old, from student to professor, to try their math skills at large. The first ones were held in the 1960s in Oxford and soon the phenomenon spread all over the world. The gatherings connect people from industry with academics for the benefit of both. Chris Budd, an enthusiast for over 20 years, tells about this exciting field. This article appeared earlier in the February 2008 issue of Mathematics Today.

What is *industrial mathematics*? Or indeed more generally what is *applied mathematics*? One view, commonly held amongst many 'pure' mathematicians is that applied maths is what you do if you can't do 'real' mathematics and that industrial mathematics is more (or indeed less) of the same only in this case you do it for money. My own view is very different from this (including the money aspect). Applied mathematics is essentially a two way process. It is the business of applying really good mathematics to problems that arise in the real world (or as close an approximation to the real world as you think it is possible to get). It constantly amazes me that this process works at all, and yet it does. Abstract ideas developed for their own sake turn out to have immensely important applications, which is one (but not the only) reason for strongly supporting abstract research. However, just as importantly, applied mathematics is the process of learning *new maths* from problems *motivated by applications*. Anyone who doubts this should ponder how many hugely important mathematical ideas have come from studying applications, varying from calculus and Fourier analysis to nonlinear dynamics and cryptography. It is certainly true that nature has a way of fighting back whenever you try to use maths to understand it, and the better the problem the more that it fights back at you! To solve even seemingly innocuous problems in the real world can often take (and lead to the creation) of very powerful mathematical ideas. Calculus is the perfect example of this. The beauty of the whole business is the way that these same ideas can take on a life of their own, and find applications in fields very different from the one that stimulated them in the first place. This process of applying mathematics in as many ways as possible can change the world. A wonderful example of this is the discovery of radio waves by Maxwell. Here a piece of essentially pure mathematics led to a whole technology which has totally transformed the world in which we live. Imagine a world without TV, radio, mobile phones and the Internet. But that is where we would be without mathematics.

So, how does industrial maths fit into the above? It is still hard to define exactly what industrial maths means, but as far as I'm concerned it's the maths that I do when I work with organisations that are not universities. This certainly includes 'traditional' industry, but (splendidly) it also includes the Met Office, sport, air traffic control, the forensic service, zoos, hospitals, Air Sea rescue organisations, broadcasting companies and local education authorities. I will use the word *industry* to mean all of these and more. Even traditional industry contains a huge variety of different areas ranging from textiles to telecommunications, space to food and from power generation to financial products. What is exciting is that all of these organisations have interesting problems and that a huge number of these problems can be attacked by using a mathematical approach. Indeed, applying the basic principles I described above, not only can the same mathematics be used in many different industries (for example the mathematics of heat transfer is also highly relevant to the finance industry) but by tackling these problems we can learn lots of new maths in the process. It is certainly true that many industrial problems involve routine mathematics and, yes, money is often involved, but this is not the reason that I enjoy working with industry. Much more importantly, tackling industrial problems requires you to *think out of the box* and to take on challenges far removed from traditional topics taught in applied mathematics courses. The result is (hopefully) new mathematics. I think it is fair to say that significant areas of my own research have followed directly from tackling industrial pro-

Figure 1 A scramble crossing in Japan. The behaviour of the pedestrians in this crossing can be analysed by using the theory of discontinuous dynamical systems.

blems. An example of this is my interest in discontinuous dynamics: the study of dynamical systems in which the (usual) smoothness assumption for these systems is removed. Discontinuous dynamics is an immensely rich area of study with many deep mathematical structures such as new types of bifurcation (grazing, border-collisions, corner-collisions), chaotic behaviour and novel routes to chaos such as period-adding. It also has many applications to problems as diverse as impact, friction, switching, rattling, earthquakes, the firing of neurons and the behaviour of crowds of people (see illustration). Studying both the theory and the applications has kept me, my PhD students and numerous collaborators and colleagues busy for many years. However, for me at least, and for many others, the way into discontinuous dynamics came through an industrial application. In my case it was trying to understand the rattling behaviour of loosely fitting boiler tubes. Trying to solve this problem by using the 'usual' theory of smooth dynamical systems quickly ran into trouble as it became apparent that the phenomena that were being observed were quite different from those predicted by the text books. Trying to address this problem immediately forced us to look at non-smooth dynamics (inspired, I should say, by some brilliant theoretical work by the industrialist who was interested in the problem and was delighted to find an academic they could talk to). However the way into this fascinating field could equally well have been a problem in power transmission in a car or the motion of buildings in an earthquake. The point is that an interesting industrial problem, far from just being an excuse to use cheap and dirty maths to make money, has in contrast led to some very exciting new mathematical ideas with many novel applications.

I must confess that I find this constant need to rise to a mathematical challenge to solve an industrial problem intensely stimulating and it continues to act as a driver for much of my research (although I don't neglect the 'pure' aspects of my research as both are needed to be able to do good mathematics). Presently I'll develop this a bit further through a couple of case studies.

How does industrial mathematics work?

Working at the interface of academia and industry is, and continues to be, a constant conflict of interest. Industry, quite rightly, has to concentrate on short term results, obtained against deadlines and may well want the second best answer *tomorrow*, rather than the best answer in a year (or indeed never). The picture I described in the previous section, of the beautiful development of a mathematical theory stimulated by industrial research, may cut little ice to the manager that needs an answer tomorrow. (Indeed, to be quite honest with you, whilst I now know vastly more about discontinuous dynamics than when I first started to look at the problem, I still cannot solve the original question of the boiler tube which turns out to be immensely difficult!). In contrast, the average academic is under intense pressure to publish scholarly research in leading journals, to develop long term research projects and to work with PhD students who often require a lengthy period of training before they are up to (mathematical) speed. This seems, at first sight, to be the exact opposite of the requirements of industry. It can, in fact, be very hard to persuade some of our colleagues on grant review panels that it is worth investing in industrial maths at all; the argument being that if it were any good then industry would pay for it, and if it is not any good then it doesn't deserve a grant! Indeed it gets worse, whilst the life blood of academics is publishing results in the open literature; industry is often constrained (quite reasonably) by the need for strict confidentiality. At first there would appear to be no middle way in which both parties are satisfied. Fortunately however, there are a number of ways forward in which it is possible to satisfy both parties. Perhaps the best of these are the celebrated *Study Groups with Industry* founded in the 1960's by Alan Tayler CBE and John Ockendon FRS. The format of a typical study group is rather like a cross between a learned conference and a paintballing tournament. It lasts a week. On the first day around eight industrial problems are presented by industrialists themselves. Academics then work in teams for a week to try to solve the problems. On the last day the results of their work are presented to an audience of the industrialists and the academics on the other teams. Does this method always lead to a problem being cracked? Sometimes the answer is yes, but this usually means that the problem has limited scientific value. Much more value to both sides are problems that lead both to new ideas and new, and long lasting collaborations, taking both academics and industrialists in completely new directions. The discontinuous dynamics example above was in fact brought to just such a meeting in Edinburgh in 1988. Another vital aspect of the Study Groups is the training that it gives to PhD students (not to mention older academics) in the skills of mathematical modelling, working in teams and of developing effective computer code under severe time pressure. It is remarkable what can be done in the hot house atmosphere of the Study Groups. As a way of stimulating progress in industrial applied mathematics, the Study Groups have no equal. The model which started in Oxford has now been copied all over the world. I have personally been attending such Study Groups (again all round the world) since 1984 and have worked on a remarkable variety of problems including: microwave cooking, land mine detection, overheating fish tanks, fluorescent light tubes, fridges, aircraft fuel tanks, electric arcs, air traffic control, air sea rescue, weather forecasting, boiler tubes and image processing, to name just a few. Of course one week is not long enough to establish a viable collaboration with industry and it is worth considering some other effective ways of linking academia to industry. A key development has been the foundation in 1987 of ECMI, the European Consortium for Mathematics in Industry by a group of active industrial mathematicians including John and Hilary Ockendon, Alan Tayler (Oxford), Sean McKee (Strathclyde) and Helmut Neunzert (Kaiserslautern) amongst others. ECMI has acted to coordinate industrial mathematics across Europe, so that the European Study Groups with Industry (ESGI) take place in different European countries throughout the year, as part of a carefully managed programme of industrial engagement with academia, which also includes regular conferences and reports on success stories. The study groups are supplemented with training camps in which students from all over Europe work together to learn the techniques for mathematical modelling. Indeed the training of PhD students in the hands on techniques to solve industrial mathematics problems is one of the best features of the study groups. The importance of involving students in industrial mathematics cannot be over emphasised, both in terms of the training that they get, and the originality that they bring to the business of solving problems. Indeed, one of my favourite mechanisms for linking academia to industrial problems is through the use of MSc projects. For seven years at Bath we have been running an MSc programme in *Modern Applications* of Mathematics which has been designed to have very close links with industry. All of the students on this course do a short three month project which is often linked to industry, with both an academic and an industrial supervisor. The nice feature about this system is that everyone wins. First and foremost, the students have an interesting project to work on. Secondly we are able to work together with industry on a project with something more closely approximating an industrial timescale and with little real risk of anything seriously going wrong. Thirdly (and to my mind perhaps most importantly) the MSc project can easily lead on to a much more substantial project, such as a PhD project, with the student hitting the ground running at the start. Ideas for such MSc projects may well come from previous Study Groups, from the Industrial Advisory Board of the MSc or (in a recent development) from the splendid Knowledge Transfer Partnership (KTP) in Industrial Applied Mathematics coordinated by the Smith Institute. The KTP is partly funded by the DTI and part by EPSRC and exists to establish, and maintain, links between academia and industry. Similar organisations such as MITACS in Canada or MACSI in the Republic of Ireland, have closely related missions.

Where is industrial mathematics going (or leading us)?

I think it fair to say that some of the great driving forces of 20th Century mathematics have been physics, engineering and latterly biology. This has been a great stimulus for mathematical developments in partial differential equations, dynamical systems, operator theory, functional analysis, numerical analysis, fluid mechanics, solid mechanics, reaction diffusion systems, signal processing and inverse theory to name just a few areas. Many of the problems that have arisen in these fields, especially the 'traditional' industrial mathematics problems (and certainly anything involving fluids or solids) are *continuum problems* described by *deterministic differential equations*. Amongst the variety of techniques that have been used to solve such equations (that is to find out what the answer looks like as opposed to just proving existence and uniqueness) are simple analytical methods such as separation of variables, approximate and formal asymptotic approaches, phase plane analysis, numerical methods for ODEs and PDEs such as finite element or finite volume methods, the calculus of variations and transform methods such as the Fourier and Laplace transforms. (It is worth noting that formal asymptotic methods have long regarded as somewhat second rate methods to be used to find rough answers rather then wait till 'proper' maths did the job correctly. However, stimulated in part by the need to address very challenging applied problems, asymptotic methods have become an area of intense mathematical study, especially the area of exponential asymptotics which looks at problems in which classical asymptotic expansions have to be continued to all orders so that exponentially small — but still very significant — effects can be resolved.) One of the consequences of concentrating on continuous problems described by PDES is that applied mathematics has, for some considerable time, been almost synonymous with fluid or solid mechanics. Whilst these are great subjects of extreme importance, and are central to 'traditional' industries involving problems with heat and mass transfer, they only represent a fraction of the areas that mathematics can be applied to. Here I believe we may see industry driving the agenda of many of possible developments of 21st Century mathematics in a very positive and exciting way. At the risk of making a fool of myself and gazing into the future with too much abandon, I think that the key drivers of mathematics will be problems dealing with *information* (such as genetics, bio-informatics and, of course the growth of the Internet and related systems), problems involving *complexity* in some form (such as problems on many scales with many connecting components and with some form of network describing how the components interact with each other), and problems centred not so much in the traditional industries but in areas such as retail and commerce. To address such problems we must move away from 'traditional' applied mathematics and instead look at the mathematics of discrete systems, systems with huge complexity and systems which are very likely to have a large stochastic component. We will also have to deal with the very difficult issues of how to optimise such systems and to deal with the increasingly large computations that will have to be done on them. One reason that I believe this is that I have seen it happening before my eyes. I have had the privilege (or have been foolish enough) to organise three Study Groups. The first of these, in 1992, had every problem (with one exception) posed in terms of partial differential equations. In contrast the Study Group I organised in Bath in 2006 had ten problems. Of these precisely one involved partial differential equations, one other involved ordinary differential equations. All the rest were a mixture of (dis-

Figure 2 There is lots of maths in chocolate, and it tastes good too!

crete) optimisation, complexity, network theory, discrete geometry, statistics and neural networks. I have seen similar trends in other Study Groups around the world. At a discussion forum at the Bath Study Group the general feeling was one of excitement (mixed with apprehension it must be said) that industry was prepared to bring such challenging problems to be looked at by mathematicians. Personally I greatly welcome this challenge. I also welcome the fact that much of the mathematics that needs to be used and developed to solve these sort of problems is mathematics that has often been thought of as very pure. Two obvious examples are number theory which comes into its own when we have to deal with discrete information (witness the major application of number theory to cryptography) and graph theory which lies at the heart of our understanding of networks. As someone that calls themselves a mathematician (rather than a pure mathematician or an applied mathematician) I strongly welcome these developments. Of course, the new directions that industry is taking applied mathematics pose interesting, and equally challenging, questions about how we should train the 'applied mathematicians' of the future. It is clear to me (at least) that any such training should certainly include discrete mathematics, large scale computing and methods for stochastic problems. How we do this is of course another matter.

Figure 3 A domestic mode-stirred microwave oven with four temperature probes used to test the predictions of the model

Some case studies

I thought that it would now be appropriate to flesh out the rather general comments above, by looking at a couple of examples. The first is (mainly) an example of a continuum problem whilst the second has a more discrete flavour to it.

Case Study one: Maths can help you to eat

One of the nicest (well certainly it tastes nice) applications of mathematics (well in my opinion at least) arises in the food industry. The food industry takes food from farm to fork, after that it's up to you. Food has to be grown, stored, frozen, defrosted, boiled, transported (possibly when frozen), manufactured, packaged, sorted, marketed, sold to the customer, tested for freshness, cooked, heated, eaten, melted in the mouth (in the case of chocolate) and digested. Nearly all of these stages must be handled very carefully if the food is going to be safe, nutritious and cheap for the customer to eat. It is very easy to think of working on food as a rather trivial application of mathematics, but we must remember that not only is the food industry one of the biggest sources of income to the UK, but also that we all eat food, it affects all of our lives and mistakes in producing food can very quickly make a lot of people very ill. Trivial it is not. Food is also a source of some wonderful mathematical problems, and (a point I take very seriously) the application of maths to food is a splendid way of enthusing young people into the importance of maths in general (especially if you bring free samples along with you!) Some of this maths is very 'traditional' applied maths much of the issues in dealing with food involve classical problems of fluid flow (usually non-Newtonian), solid mechanics (both elastic and visco-elastic), heat transfer, two phase flow, population dynamics (such as fish populations) and free boundary problems. Chocolate manufacture for example, involves very delicate heat flow calculations when manufacturing such delicate items as soft centre chocolates However problems involving the packaging, marketing, distribution and sale of food lie more properly in the realms of optimisation and discrete mathematics. It is clear that the food industry will act as a source of excellent mathematical problems for a very long time to come.

Two problems, in particular, that I have worked on concern the micro-wave cooking and the digestion of food. For the sake of the readers sensitivities I shall only describe the former in any detail, although it is worth saying that modelling digestion is a fascinating exercise in calculating the (chaotic) mixing of nutrients in a highly viscous Non-Newtonian flow driven by pressure gradients and peristaltic motion and with uncertain boundary conditions. As any student knows, a popular way of cooking (or at least of heating) food is to use a micro-wave cooker. In such a cooker, microwaves are generated by a Magnetron (also used in Radar sets) and enter the oven cavity via a waveguide or an antenna. An electric field is then set up inside the oven which irradiates any food placed there. The microwaves penetrate the food and change the orientation of the dipoles in the moist part of the food leading to heating (via friction) of the foodstuff and consequent phase changes.

A problem with this process is that the field can have standing wave patterns, which can result in localised 'cold-spots' where the field is relatively weak. If the food is placed in a cold-spot then its temperature may be lower there and it will be poorly cooked (see figure). To try to avoid this problem the food can either be rotated through the field on a turntable, or the field itself can be 'stirred' by using a rotating metal fan to break up the field patterns.

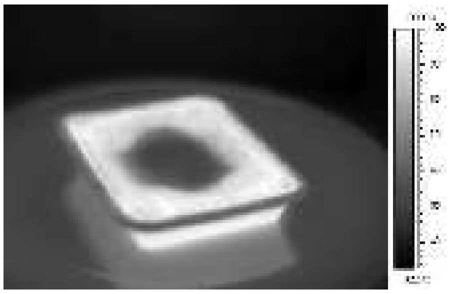

Figure 4 Thermal camera image of food in a domestic turntable oven, showing a distinct 'cold spot' in the centre caused by a local minimum in the radiant electro-magnetic field

An interesting 'industrial mathematics' problem is to model the process by which the food is heated in the oven and to compare the effectiveness of the turntable and mode-stirred designs of the micro-wave oven in heating a moist foodstuff. This problem came to me through the KTN and a Study Group and was 'sub-contracted' to a PhD CASE student Andrew Hill. One way to approach it is to do a full three dimensional field simulation by solving Maxwell's equations, and to then use this to find the temperature by solving the porous medium equations for a two-phase material. The problems with this approach are (i) the computations take a very long time, making it difficult to see the effects of varying the parameters in the problem (ii) it gives little direct insight into the process and the way that it depends upon the parameters and (iii) micro-wave cooking (especially the field distribution) is very sensitive to small changes in the geometry of the cavity, the shape and type of the food and even the humidity of the air. This means that any one calculation may not necessarily give an accurate representation of the electric field of any particular micro-wave oven on any particular day. What is more useful is a representative calculation of the ave-

Figure 5 Three trip-wires are hidden in this image. Can you find them?

rage behaviour of a broad class of (domestic) micro-wave ovens in which the effects of varying the various parameters is more transparent. Here a combination of both an analytical and a numerical calculation proved effective. In this calculation we used a formal asymptotic theory both to calculate an averaged field and to determine how it penetrated inside a moist foodstuff. (Note, contrary to popular myth, microwaves do not cook food from the inside. Instead they penetrate from the outside, and if the food is too large then the interior can receive almost no micro-wave energy, and as a result little direct heating. This is why the manufacturers of micro-wave cookable foods generally insist that, after a period of heating in the oven, the food is stirred to ensure that it is all at a similar temperature.) The temperature T of the food satisfies the equation $u_t = k\nabla^2 T + P(x, y, z, t)$ where u is the enthalpy and P is the power transferred from the microwave field to the food. As remarked above, finding P exactly was very hard, however a good approximation could be found asymptotically (in particular by using the WKBJ method) for ovens with either a mode-stirrer or a turntable. This approximation showed that the overall the field decayed exponentially as it penetrated the food but that on top of this decay was superimposed an oscillatory contribution (due to reflexions of the radiation within the food) the size of which depended upon the dimensions of the food. A relatively simple calculation showed that these oscillations were small provided that the smallest dimension of the food was larger than about 2cm. Fortunately, most foodstuffs satisfy this condition. As a result it was possible to use a much simpler description of the electric field in the enthalpy equation than that given by a full solution of Maxwell's equations, and numerical approximations to the solution of these simplified equations were found very quickly on a desktop PC. When compared against experimental values of both the temperature and the moisture content these solutions were surprisingly accurate, given the approximations that are made, and gave confidence in the use of the model for further design calculations. It was the combination of mathematics, numerical methods, physical modelling and the careful use of experimental data that made this whole approach successful and is typical of the mix of ideas that have to be combined to do effective industrial mathematics.

Case Study Two: Maths Can Save Your Life

One of my favourite 'industrial mathematics' problems came up in a recent study group, and is an example of an application of signal processing and information theory which can potentially save peoples lives. One of the nastiest aspects of the modern world is the existence of anti-personnel land mines. These unpleasant devices, when detonated, jump up into the air and kill anyone close by. They are typically triggered by trip-wires which are attached to the detonators. If someone catches their foot on a trip wire then the mine is detonated and the person dies. To make things a lot worse, the land mines are typically hidden in dense foliage and thin nylon fishing line is used to make the trip wires almost invisible. One way to detect the land mines is to look for the trip wires them selves. However, the foliage either hides the trip-wires, or leaf stems can even resemble a trip wire. Any detection algorithm must work quickly, detect trip-wires

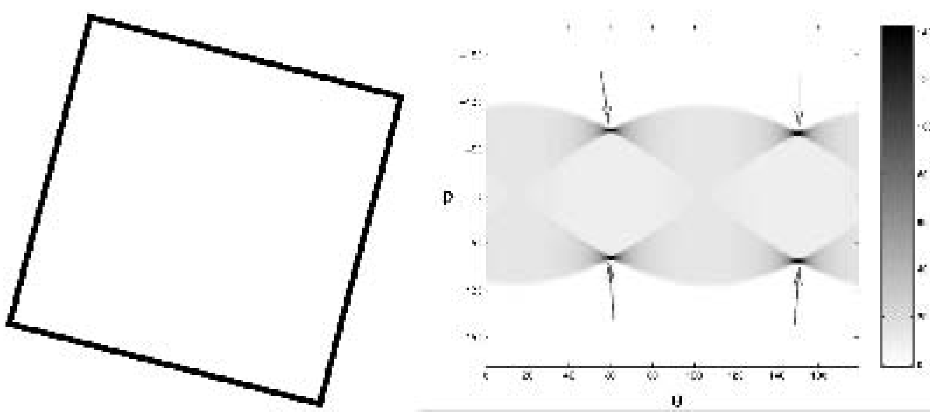

Figure 6 Left: a square; right: the Radon Transform of the square on the left. The four straight lines that make up the sides of the square show up as points of high density.

when they exist and not get confused by finding leaves. An example of the problem that such an algorithm has to face is given in figure 5 in which some trip-wires are hidden in an artificial jungle.

In order to detect the trip wires we must find a way of finding partly obscured straight lines in an image. Fortunately, just such a method exists; it is the Radon Transform (or its various discrete versions). In this transform, the line integral $R(\theta,\rho)$ of an image with intensity $u(x,y)$ is computed along a line at an angle θ and a distance ρ from the centre of the image so that

$$R(\theta,\rho)=\int u(\rho\cos(\theta)-s\sin(\theta),\\ \rho\sin(\theta)+s\cos(\theta))\,ds.$$

This transformation lies at the heart of the CAT scanners used in medical image processing and other applications as it is closely related to the formula for the attenuation of an X-Ray (which is long and straight, like a trip-wire) as it passes through a medium of variable density (such as a human body). Indeed, finding (quickly) the inverse to the Radon Transform of a (potentially noisy) image is one of the key problems of modern image processing. It has countless applications, from detecting tumours in the brain of a patient (and hence saving your life that way), to finding out what (or who) killed King Tutankhamen. For the problem of finding the trip-wires we don't need to find the inverse, instead we can apply the Radon transform directly to the image. In figure 6 see on the left a square and on the right its Radon Transform.

The key point to note in these two images is that the four straight lines making up the sides of the square show up as points of high intensity (arrowed) in the Radon Transform and we can easily read off their orientations. Basically the Radon Transform is good at finding straight lines which is just what we need to detect the trip-wires. Of course life isn't quite as simple as this for real images of trip-wires and some extra work has to be done to detect them. In order to apply the Radon transform the image must first be pre-processed (using a Laplacian filter and an edge detector) to enhance any edges. Following the application of the transform to the enhanced image a threshold must then applied to the resulting values to distinguish between true straight lines caused by trip wires (corresponding to large values of R) and false lines caused by short leaf stems (for which R is not quite as large). However, following a sequence of calibration calculations and analytical estimates with a number of different images, it was possible to derive a fast algorithm which detected the trip-wires by first filtering the image, then applying the Radon Transform, then applying a threshold and then applying the inverse Radon Transform. (The beauty of this is that most of these algorithms are present in the MATLAB Signal Processing Toolbox. Indeed, I consider MATLAB to be one of the greatest tools available to the industrial mathematician.) The result of applying this method to the previous image is given in figure 7 in which the three detected trip-wires are highlighted.

Figure 7 The three trip wires detected using the Radon Transform

Note how the method has not only detected the trip-wires, but, from the width of the lines, an indication is given of the reliability of the calculation. All in all this problem is a very nice combination of analysis and computation.

Signal processing problems of this form are not typically taught in a typical applied mathematics undergraduate course. This is not only a shame, but denies the students on those courses the opportunity to see a major application of mathematics to modern technology.

Conclusions

I hope that I have managed to convey some of the flavour of industrial mathematics as I see it. Far from being a subject of limited academic value, only done for money, industrial maths presents a vibrant intellectual challenge with limitless opportunities for growth and development. This poses significant challenges for the future, not least in the way that we train the next generation of students to prepare them for the very exciting ways that maths will be applied in the future, and the new maths that we will learn from these applications.

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History

The Royal Dutch Mathematical Society since 1778

The Royal Dutch Mathematical Society and its associated journal the (Nieuw) Archief voor Wiskunde enjoys an interesting history with a lot of intrigues. Danny Beckers, historian of mathematics at the VU University, Amsterdam, highlights four important turning points of its history. Retrospectively, the original motto of the society, 'untiring labour overcomes all', has been chosen very accurately. Although at the time of inception, this motto referred to the way out of economic depression, as well as to the mental process of acquiring mathematical knowledge, in time it also came to represent the work of the mathematician on behalf of the Dutch mathematical society.

This is the story of a small-scale Amsterdam initiative that grew into the *Koninklijk Wiskundig Genootschap* (Royal Dutch Mathematical Society). It is a story of four people, who all had a profound influence on the society and who shared a passion for mathematics. Acting in favour of the society, in pursuit of their private goals or both, they grasped the opportunities that came their way, and by doing so they passionately contributed to the shape of the *Koninklijk Wiskundig Genootschap*, the society that now publishes this journal and is offering its patronage to the 5th European Congress of Mathematics.

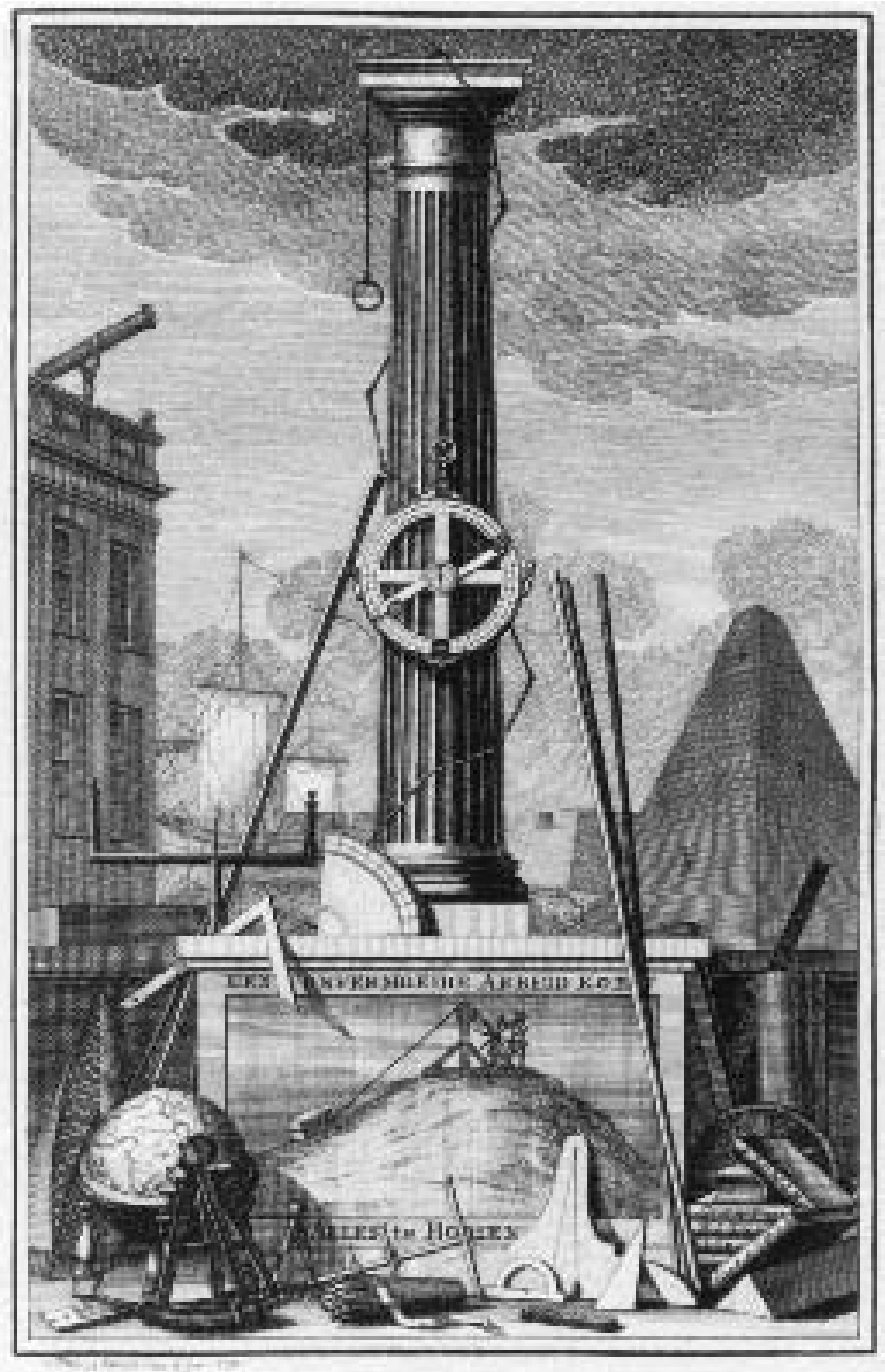
A pastime for amateurs

During the 18th century the publication of journals and books bore witness to a growing interest in mathematics as a pastime in the Netherlands. British coffee houses and pubs of the time flourished as meeting places of mathematical amateurs, each trying to amuse each other with their wit — put to the test by exercises in elementary geometry and algebra. In the Netherlands similar social gathering took shape comparatively late, in the 1770s and 1780s, when various local mathematical societies were founded. In these times of political turmoil the cultivation of mathematics was all part of a commitment to an enlightening of society: the practice of mathematics was held as beneficiary to Dutch culture and economy.

A journal as motivation

In 1770 Arnold Strabbe (1741–1805), a well-known figure in the Amsterdam book trade, succeeded in convincing one of his publishers to start a mathematical journal. As the journal was discontinued within two years, Strabbe realised that financing such a journal would always remain problematic. Funding publications through a society, like the Hamburg Mathematical Society of which he was a member, would offer a much better guarantee for continuity. By 1778 he had gathered enough interested parties to found the Amsterdam *Wiskundig Genootschap*, which chose as its motto "untiring labour overcomes all". The society thrived on Strabbe's efforts and network. He was collecting exercises and translating books on behalf of the society. A journal was issued at irregular intervals. However, as the society grew, Strabbe was found to be favouring his own material, using the society's funding as a promotion vehicle for his own books and translations. The resulting clash with some of the younger society members culminated in a dramatic attack on his person. Strabbe was kicked out of office in 1804 and died still holding a grudge only a year later.

If Strabbe may be considered the founding father, it was Jacob de Gelder (1765–1848), one of the 'young Turks' who took over, who turned a local initiative into a truly national society. On the wings of an emerging national education policy, De Gelder exerted his influence by focusing on standards for good mathematics education in the Netherlands. Even after he had left Amsterdam in pursuit of his career, de Gelder continued to act on behalf of the society. As he made his living by publishing textbooks, he simultaneously set a standard for good and thorough mathematics education. Through the influence of his writings, promoting mathematics as the most suitable route towards unity and prosperity in the country, and by making good use of his political connections, an official goal of the mathematical society was reached. Mathematics became an obligatory part of the secondary school curriculum, most notably contesting Latin and Greek as a way of acquiring true knowledge. Internally the *Genootschap* thrived on the efforts of amateurs and teachers. As in Strabbe's days it published textbooks, exercises and papers at irregular intervals.



Frontispiece of *Kunstoefeningen over verscheide nuttige onderwerpen der wiskunde*, 1 (1782)
This is the first publication of the Wiskundig Genootschap. This frontispiece of the journal of the 'Wiskundig genootschap' illustrates the concerns of the original members of the society. Central to the picture is a huge pilar, representing architecture. Wrapped around the pillar is a surveyors chain and right in the centre of the whole picture is a Borda circle — used for measuring lunar positions, in order to establish longitude at sea.
Other instruments are shown, such as a quadrant, a globe, a Jacob's staff, a sextant, a compass, measuring sticks, and a telescope (on the top of the building to the left). The ship and the fortress in the background represent navigation and fortress building which were important fields of work for the members of the society. The pile of books in the lower right corner read (a.o.) the names of Euclid, Newton and Metius. The books represent the canonical literature of mathematics.
The pyramid deserves our special attention. The building originated from the emblem of the society, where it represented the motto: untiring labour overcomes all. In the emblem it was climbed by several people —representing the untiring labour— while one man was standing at the top, raising his hands in exaltation: he showed that untiring labour really overcame all. In this frontispiece the pyramid is more in the background, and the motto is around the painting in front (the text on the pedestal means 'Untiring labour overcomes all'), where a man is hauling a huge stone up a hill, using a mechanical device.

Professionalism

Among the amateur mathematical societies in Europe the Dutch *Wiskundig Genootschap* was a late entrant. This gives it an early appearance among the professionalisation societies of mathematicians, typical of the second half of the nineteenth century. From 1867 onwards, national mathematical societies were being founded in most European countries. These served as a vehicle for professionalisation for a growing group of university mathematicians, creating a national as well as an international stage. David Bierens de Haan (1822–1895) led the *Genootschap* to its first step towards a mathematical society in this modern sense. Bierens de Haan, an internationally well-connected mathematician, was aware of the backwardness of the society and its publication policy. When asked to take up the editorship of the society's journal, he insisted on modernising the journal by focusing on shorter papers on relevant new theories and proofs, issuing at regular intervals with a steady editorship. In order to emphasize the novelty, the journal, published since 1875, was renamed *Nieuw Archief voor Wiskunde* (New Archive for Mathematics), creating a clear distinction from the old *Archief voor Wiskunde* and its predecessors. No longer was the journal being published solely on behalf of the society's members. Addressing all mathematicians, it modernised not only its content but its envisaged readership as well.

International aspiration

The international stage was legitimised by emphasizing the universality of mathematics. Bierens de Haan actively represented the Netherlands in international efforts to canonise the international literature and to write the Dutch history of mathematics, as a notable part of the history of mathematics as a whole. The international stage existed in journals and increasingly in meetings, growing into conferences and growing into congresses.

At the mathematics section of the 1889 international congress in Paris, it was agreed that a system to catalogue all publications in mathematics and a reference journal based on it was desirable; Bierens de Haan was closely involved and took the challenge home to Amsterdam.

It was under the presidency of Diederik Johannes Korteweg (1848–1941) that the *Wiskundig Genootschap* was able to answer the challenge and thereby earned its place among the mathematical societies of the world. In 1892 the *Genootschap* received a

Jacob de Gelder

David Bierens de Haan

Diederik Johannes Korteweg

large bequest. Korteweg, together with his friend Pieter Hendrik Schoute (1846–1913), realised that these funds would allow the society to issue such a journal of abstracts. In a further Paris meeting in 1892 the society was formally assigned the task and the *Revue sémestrielle* was published from 1893 to 1930. The enterprise put Dutch mathematics in the picture, internationally and at home, by involving every able mathematician in the abstracting work. The work on behalf of the *Revue* created a core group of mathematicians dedicated to research, who were aware of the foreign literature and were embedded in an international network of colleagues. These efforts were symbolic of the changing character of the *Wiskundig Genootschap*. The society had served a broad audience of mathematicians, from interested laymen to teachers and from scientists to actuaries. In the early twentieth century, it reinvented itself to emerge as a research oriented society. The *Nieuw Archief voor Wiskunde* became a research journal and was distributed internationally and largely lost the interest of the Dutch mathematics teachers and other non-scientists. To keep in touch with 'the mathematicians in the field' the society continued to publish exercises, but by 1924 the teachers had established their own journal, *Euclides*. The prime objectives of the society and its journal had definitely changed.

Informing a professional audience

In 2000, a century on from Korteweg and the turn towards a research journal, the *Nieuw Archief voor Wiskunde* returned to a magazine addressing a larger audience of those interested in mathematics. These days we do not feel such a strong urge to emphasize the discontinuity but just start a 'new series', the *Vijfde Serie* (Fifth Series) of the *Nieuw Archief*.

On 1 May 2003, nine quarters of a century on from its inception in 1778 amidst the republican turmoil, the Dutch Mathematical Society was granted the predicate 'Royal'. From that date the society has been formally named *Koninklijk Wiskundig Genootschap*. If we have restricted the above story to four people, such recent achievements and changes in policy are there to remind us that mathematicians never get tired. 'Untiring labour' turns out to have been a very relevant motto.

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Research

Forensic Statistics

Lately, forensic science has been receiving much attention in the media. Popular programs such as 'Crime Scene Investigation' show how forensic investigations can contribute to obtaining and reporting results in criminal law. These investigations cover a broad range of areas of expertise, varying from DNA profiling to examining shoe prints, and from toxicology to handwriting. Experts in other scientific fields are also regularly called upon to appear in court. Statistics and probability theory frequently underlie statements made by experts and therefore play an important role in the judicial system. Nevertheless, the fields themselves are seldom placed in the limelight. Forensic statistics is a broad and dynamic field, in which there is still much work to be done. For the mathematician it is a fascinating applied field with exciting examples. Marjan Sjerps works as a statistician at the Netherlands Forensic Institute. This text is a translation of a 2004 prize winning article from the Nieuw Archief voor Wiskunde.

According to those in the know, the role of experts in criminal law is on the rise (Nijboer, 2002). Since statistics forms an important component of many areas of expertise, it also enjoys increasing attention. Outside of the Netherlands a number of publications and websites are dedicated to statistics in (criminal) law. Examples are Finkelstein and Levin 1990, Aitken and Stoney 1991, Gastwirth 2000, Good 2001, websites of the Federal Judicial Centre and of Charles Brenner. There are only a few references in Dutch; see for example van Koppen et al. 2002 and Broeders 2003, which mentions it briefly. In 2005 different authors contributed to a book (Sjerps and Coster van Voorhout). Since very diverse areas of expertise such as pathology, biology, physics, chemistry and psychology are involved, the statistical techniques that are used are also very diverse. Nevertheless, these areas have a number of aspects in common when applied to criminal law. For example, there is usually a suspect and a chosen characteristic of this suspect is compared to trace evidence associated to the crime. The DNA profile of the suspect can be compared to the DNA profile of a trace found in the underwear of a victim of sexual abuse (see figure 1), or a rifle found in the suspect's house can be compared to a bullet found at the scene of an armed robbery. In another case a dog may compare the suspect's scent to a scent that is thought to be on an object touched by the perpetrator, after which the expert must report the evidential value of the dog indicating (or not) a similarity between the scents. In this type of research, the important factor is the evidential value of observed similarities and differences. Up to approximately twenty years ago, there was no general usable definition of the term *evidential value* and every area of expertise had its own method to reflect the evidential value of its research results. This void was filled by the emergence of the so-called Bayesian approach in forensic statistics (also called the 'logical' or 'likelihood ratio' approach). A model was developed that was based on a well-known rule of Bayes from probability theory. It gives a quantitative expression for the evidential value of different types of evidence but can also be used as a guiding principle for experts reporting their conclusion verbally. The standard texts in this field are Aitken (1995) and Robertson and Vignaux (1995). This approach is applied to an increasing number of subfields, sometimes giving rise to interesting questions (see the frames). In this article we will briefly explain the Bayesian model. We will also discuss three pitfalls in which the average lawyer easily falls when interpreting evidence, and many mathematicians with

Formulating hypotheses
A still unsolved problem concerning the foundations of statistics is the following. Suppose that at the site of a crime committed by two perpetrators two blood traces A and B are found, possibly originating from the two perpetrators. The blood traces are tiny and of terrible quality, making only a partial DNA-profile possible. The profiles show that the two blood traces were made by different persons. The 'match probability' is one in a million for trace A and one in a hundred for trace B. The police catch one suspect, whose DNA profile corresponds to that of trace A. The evidential value, measured by the *likelihood ratio* (LR), now depends on the wording of the hypotheses. The LR for the two hypotheses (1a) The suspect is the donor of one of the two traces (1b) The suspect is not the donor of either of the two traces is half a million; the LR for the two hypotheses (2a) The suspect is the donor of trace A (2b) The suspect is not the donor of trace A is one million. One difference between the pairs of hypotheses is that the first pair can be stated without knowing the DNA profiles of the traces and of the suspect, while the second is based on this information. In both cases the computation of the LR is based on the assumption that the traces A and B were made by different persons, but this is only known after DNA profiling is carried out. The discussion now centres on the following. May the expert formulate his hypotheses using the available evidence and then present the LR for these hypotheses? Or is it illegal to use available evidence because the statements of the hypotheses should be independent of the evidence (see also Meester and Sjerps 2004 and the discussion with Dawid).

him. See Broeders (2003) for a fairly complete overview of this type of pitfall. A completely different branch of forensic statistics concentrates on the way data is collected. This can for example concern environmental crime, with questions like how to sample a landfill, or barrels containing XTC production waste. Designing experiments for investigations also belongs to this subfield. I will touch upon this briefly. To indicate why forensic statistics is more than just the application of ordinary statistics to an unusual field, I will also briefly discuss the specific forensic aspects. Finally, I will look beyond my own sphere of activities, the statistical aspects of the technical forensic research done by the Netherlands Forensic Institute, and briefly mention some other areas in which statistics is used for criminal law.

How strong is the evidence: the Bayesian model
To express the value of evidence, the Bayesian approach in forensic statistics uses a general model based on Bayes' theorem:

$$P[A|B] = \frac{P[B|A] \cdot P[A]}{P[B]} .$$

In this expression, $P[A]$ is the probability of A being true and $P[A|B]$ is the conditional probability of A being true given that B is true. If we apply this rule to two hypotheses H_p and H_d and evidence E, we can deduce (using a subjective definition of probability) that

$$\frac{P[H_p|E]}{P[H_d|E]} = \frac{P[H_p]}{P[H_d]} \cdot \frac{P[E|H_p]}{P[E|H_d]} .$$

We interpret this formula as meaning that the ratio of the probability of two hypotheses is modified by introducing evidence. The new odds (on the left) arise from the old odds (in the middle) by multiplication by the *likelihood ratio* LR (the ratio on the right). In words: *the posterior odds equal the prior odds times the likelihood ratio.*

In criminal law this rule can be applied as follows: the prosecutor has a hypothesis H_p, for example 'the suspect wrote this threatening letter', and the defender has another hypothesis H_d, for example 'someone else wrote the letter'. The judge will convict the suspect if the prosecutor's hypothesis is much more probable than the defender's, that is, if the posterior odds of these two hypotheses are large. At the start of the trial, the judge has a certain estimate of the prior odds. During the trial, evidence is brought forward: some of it in favour of the suspect and some of it against. Every time a piece of evidence is brought forward, the judge will adjust his estimate of the probability ratio according to Bayes' rule. That is, he multiplies it by an LR greater than one when the evidence is against the suspect and with a LR smaller than one when the evidence is in favour of the suspect. Once all evidence has been presented, the judge makes a decision based on his final estimate of the probability ratio between the prosecutor's hypothesis and the defender's.

Figure 1 A suspect's DNA profile is compared to the DNA profile of a trace from the victim's underwear in a case of sexual abuse.

Although this appears to be a simple application, this thought process has consequences for the way experts must report their results. The expert cannot make a pronouncement about the posterior odds of the prosecutor's and defender's hypotheses without making an assumption on the prior odds. For example, the forensic handwriting expert cannot say anything about the probability that the suspect has written the threatening letter without making assumptions about the probability before he studies the letter.

Such assumptions are pre-eminently part of the judge's tasks, not the expert's. It follows that based on his knowledge, the expert can only make a pronouncement concerning the LR of the evidence for the hypothesis H_p versus the hypothesis H_d. According to the Bayesian model, this is exactly the expert's task: reporting this LR so that the judge can subsequently adjust his estimate of the probability ratio of H_p versus H_d.

An example based on a true case
At the scene of a burglary the perpetrator cuts himself on the glass of a window that was smashed. The DNA expert determines the DNA profile of the blood on the glass, which is then compared to that of the suspect. In the courtroom the prosecutor claims that the blood comes from the suspect (H_p), while the suspect's lawyer says that his client has nothing to do with the case and that the blood is someone else's (H_d). Suppose that the DNA profiles are a perfect match; what is then the LR? The numerator is the probability of a

The value of a DNA database match

A bank is robbed by a masked person. The car that is used to leave the crime scene is dumped and set on fire. The mask is found inside. The traces of saliva found on the partially burnt mask result in a DNA profile that has a probability of one in ten thousand of matching the profile of an random person (due to the low quality of the traces only a partial DNA profile is obtained). The saliva profile is compared to a DNA database of 3000 suspects, one of whom has the same DNA profile. The trace was compared to 3000 persons instead of only one. Question: does comparison with the DNA database increase or decrease the value of the DNA match as evidence? *Answer 1*: by making several comparisons the probability of a chance hit increases. If there were ten thousand people in the database, you would already expect a match to occur simply by chance. The value of the match as evidence therefore decreases. *Answer 2*: the information that the 2999 'non-matching' persons in the database can be excluded as perpetrator increases the value of the evidence. If the whole world population were in the database and there were only a single match, we would be certain that he is the donor of the saliva. The evidential value of the match therefore increases. For the correct answer, see the frame at the end of this article.

match when the blood sample comes from the subject. If no mistakes are made, this probability is one. The denominator is the probability of a match when the blood sample comes from an random person.

Based on reference material, the expert has some knowledge of this 'match probability': it is less than one in a billion. The LR is therefore one billion, which means that the probability ratio of H_p versus H_d is increased by a factor of one billion. However, the expert cannot say how large this ratio becomes since this depends on the prior odds. This way, the Bayesian approach provides both a definition of the expert's task and a quantitative definition of the notion of 'evidential value'. Indeed, it follows from the model that the size of the LR reflects exactly how much more probable the prosecutor's hypothesis becomes with respect to the defender's when evidence is added. In other words, the LR exactly reflects the value of the evidence. In the example given above, the evidential value is at least one billion. If the blood sample were of such low quality that only a partial DNA profile could be obtained with a match probability as large as one in a million, the LR would have been a million. The Bayesian model can also be used to combine several pieces of evidence. If the hypotheses remain unchanged, it easily follows that the LR of the combination of two independent pieces of evidence is the product of the LRs of the individual pieces. This means that, in principle, when analyzing hair a morphological study can be combined with DNA profiling. In less simple cases, the analysis rapidly becomes very complex. At present, research is being done into the use of 'Bayesian belief networks' for combining evidence.

The LR approach has been applied successfully in cases where numbers are available. One example is DNA profiling (see for example Evett and Weir 1998, and Sjerps and Kloosterman 2003), where the LR is used not only for simple comparisons of a trace with a suspect but also in more complex situations where the numerator of the LR is not one. DNA testing to establish relatedness is such an area, where, for example, an unknown victim is compared with puatitve parents, or a foetus with the mother and her putative rapist. Less well-known examples are automatic speaker recognition, where a computer program is used to compare the voice of someone on a telephone tap to a suspect's voice (see Broeders 2003 and references found therein), and the comparison of glass (Curran et al. 2000). LRs can also be computed for scent identification tests with dogs, face recognition by witnesses and polygraph tests (see Van Koppen et al. 2002 and references found there; here the LR is usually referred to as the 'diagnostic value'). Even if there is some discussion concerning the formulation of hypotheses and the reporting of an LR in areas where the LR can be calculated effectively, nowadays most experts agree that it is their task to report an LR. In areas where numbers are less readily available, the Bayesian model has raised a great deal of controversy and it is not generally accepted that an expert must report an LR. For a long time, verbal statements have been made in these areas concerning the probability of the prosecutor's hypothesis and few experts are inclined to change this. For example, an fire arms expert may conclude that 'the probability that the bullet was fired by the suspect's gun is very high'. In the Bayesian model he cannot make such a conclusion based on his expertise because the probability depends on the prior odds. In Bayesian terms, the expert should state his conclusion in another way; for example: 'our investigation gives a very strong indication that the bullet came from the suspect's gun'. Few lawyers will recognize the difference between these two statements. The discussion concerning reporting is therefore still ongoing in such areas.

Fallacies in the appreciation of evidence

In addition to providing the definition of the expert's role and of evidential value, the Bayesian way of thinking has also revealed a number of fallacies. These are based on common errors of thinking and have been known for some time in psychology (Kahneman et al. 1982). Within forensic literature, however, they have only been seriously considered in the last thirteen years (Evett 1995, Broeders 2003). The most well-known is the *prosecutor's fallacy*. Suppose, for example, that the perpetrator's blood is found at a crime scene. The (partial) DNA profile is compared to that of suspect J. Smith and matches. The probability that an arbitrary person would have this profile is, let's say, one in a million. The prosecutor can present the following reasoning:

1. The probability that the profiles match while the blood is not J. Smith's is one in a million.
2. The profile matches, so the probability that the blood is not J. Smith's is one in a million.
3. The probability that the blood is J. Smith's is therefore 99.9999%.

Obviously the prosecutor makes a mistake when he concludes (2) from (1). Indeed, the probability that the samples match given that the blood is not J. Smith's is not equal to the probability that the blood is not J. Smith's given that the samples match. Nevertheless, the literature shows that this error is made by many. My own restricted experience and that of my colleagues also shows that Dutch lawyers sometimes make this mistake. The consequences can be considerable in cases where the 'match probability' is relatively large and there is barely any other evidence. In the same case, the defender can reason as follows:

1. The Netherlands has about 16 million inhabitants other than J. Smith.
2. We can expect 17 persons, including J. Smith, to have the same DNA profile.
3. The probability that the blood is J. Smith's is about 1 in 17, that is, approximately 6%.

In itself, this reasoning is correct but it is based on a number of hidden assumptions, for example that the perpetrator is Dutch and that all Dutch (including all women, babies, elderly, handicapped, etc.) have the same prior probability of being the perpetrator. This reasoning is also sometimes used in Dutch courtrooms. In the literature it is called the *defence fallacy*. Finally, there is also the *base rate fallacy*. In this case, the LR is not scaled by the prior odds. Suppose, for example, that a masked robbery takes place where the perpetrator's saliva is found on a cap lying in the car used to leave the crime scene. If there aren't any suspects yet, the expert can compare the frequency of the DNA profile of the saliva in different populations.

Suppose that in population A the frequency is one in a million and in population B it is one in a billion. The LR for the hypothesis that the perpetrator comes from population A and not B is then one thousand. The base rate fallacy is now that one tends to think that this means that the probability that the perpetrator comes from population A is much higher than that he comes from B. However, the probability that the perpetrator comes from population A of course also depends on a great number of other factors, of which the size of the population is the most obvious.

The above makes it clear that it is not sufficient for a forensic expert to decide what his statement is about, he must also consider how he can prevent fallacies and misunderstandings when formulating his conclusions. However, the ultimate solution for the prevention of fallacies and misunderstandings is yet to be found; the importance of psychological research in this is increasing.

Figure 2 Environmental investigation is an area where sampling is of great importance for the final conclusions. Up to now laboratories have always spent much time and money optimizing their analytical methods, while little attention is payed to the taking of samples. Nowadays it has been recognized that the biggest source of error is not in the lab but in the field.

Gathering data

The above concerns situations where conclusions must be made based on research results. However, a large part of the questions my colleagues and I were asked recently concerned situations where observations still had to be carried out. Environmental investigation, for example, is an area where sampling is of great importance for the final conclusions. Often the question is whether the concentration of certain substances exceeds the legal limit. Sampling and sample analysis are very expensive, while the batches that are concerned can be large and heterogeneous (see figure 2). Consequently, in the sampling protocols that are legally mandated, a large batch can be approved or rejected based on only a few ianalytical results. In the Dutch regulation concerning the sampling and analysis of building materials (execution decree 1998), for example, a 2000 tonne lot is approved or rejected based on three analytical results, each obtained by mixing four samples. In practice, it turns out that the lots do not always satisfy the assumptions, for example because illegal substances were mixed in, making the lot more heterogeneous than was assumed. Furthermore, 'hot-spots' in the lot can cause outliers during analysis. The effects of this and the correct statistical treatment are still unclear. For statisticians, this area forms an enormous challenge (see among others Keith 1996, Gilbert 1987, Patil and Rao 1994 and Stelling and Sjerps 1999). Sampling often still does not receive the attention it deserves. Laboratories spend much time and money optimizing their analytical methods, while they pay remarkably little attention to the samples they receive. In practice, the most important source of error is not in the lab but in the field. The differences between samples are usually much bigger than the differences between repeated analyses of the same sample. The legislature also pays little attention to this fact: the suspect is allowed a counter analysis in an independent laboratory but the samples used there are taken right next to the samples that were analyzed before.

In setting up a forensic investigation, one always comes across the question of how observations must be collected. An example is a case where the suspect claimed that his gun went off accidentally while he made a certain manoeuvre. The fire arms experts of the NF determined the probability of such an event through an experiment in which they repeated the suspect's manoeuvre with his weapon approximately 400 times. Many days and blisters later the experts found that the gun had not gone off a single time. In general, however, experiments are done within the framework of a research project, for example to introduce a new method. A third situation where the gathering of data is important concerns the sampling of a batch of discrete units, such as numerous bags of narcotic substances or pills, a set of barrels with unknown contents that has been dumped somewhere in an abandoned area, or a great number of fibres found on a victim's clothing, of which only a limited number can be analyzed more closely. There are different methods of sampling in such cases. At present a directive for sampling in batches of narcotic substances is being developed on a European level, which includes a discussion of the advantages and disadvantages of these methods (ENFSI Drugs WG 2003).

Specific forensic aspects

Forensic statistics is characterized by a number of particular aspects. First of all, of course, we have the multifaceted field in which it is applied, which asks for a broad range of statistical techniques, e.g. experimental design, classification, reliability intervals, Bayesian methods, simulation techniques, non-parametric methods, control charts, regression and population genetics. Moreover, the objects that are studied are not commonplace. In DNA-profiling it can, for example, concern blood, saliva or sperm traces, which are sometimes available only in very small quantities, partially decayed or mixed together. In physical investigations, it can, for example, concern garbage bags, screwdrivers, shoes (see figure 3), bullets, handwriting, voice or noise; in chemical investigations, it can be explosives, gunshot residue, glass fragments, tape, ink or car paint. Even in comparing ear prints

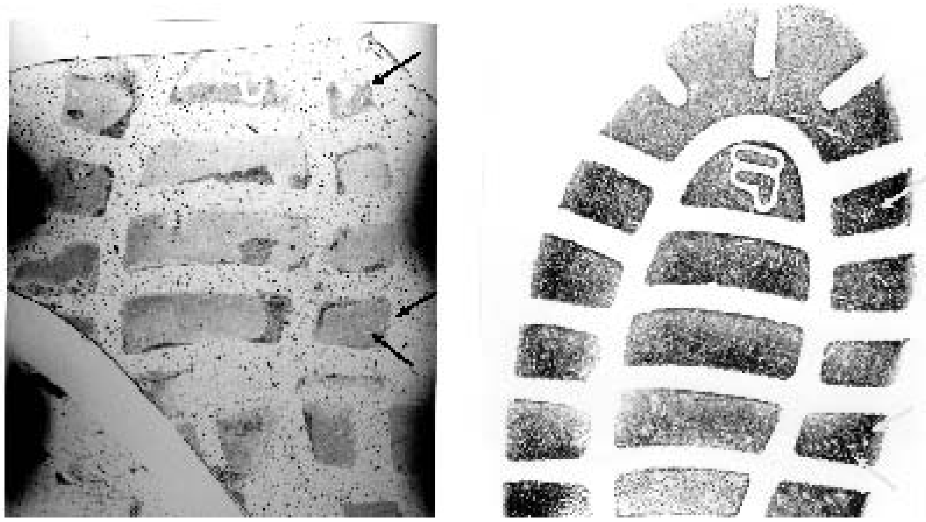

Figure 3 In legal cases it is often important to know the probability that, for example, a suspect's shoe caused a given print. Many forensic statisticians believe that the Bayesian, or subjective, notion of probability is better adapted to this type of question than the mathematical notion of probability that is based on Kolmogorov's axioms.

quite a bit of statistics is involved. Casework aimed at individualisation, where a trace is compared to a certain characteristic or object associated to the suspect, has the particular property that the attention is turned towards the relation between the trace and one specific object or individual. The lawyer, for example, wants to know the probability that this particular shoe of suspect Jones has caused the shoe print. Because of this the Bayesian or subjective notion of probability is much better suited to the situation than the frequentist or mathematical notion of probability based on Kolmogorov's axioms. In a forensic context it is very artificial to see a probability as the ratio of the number of times that Jones' shoe made the print when one hundred thousand prints are made, or as some abstract probability measure. Often the police find a number of pieces of forensic evidence in one case. At a burglary, for example, fingerprints, shoe prints, DNA traces and tool marks can be found. A specific forensic question is then not only what the evidential value of the different pieces of evidence are but also what the evidential value is of the combination of all these pieces. In this case the Bayesian model offers a directive, as described above. The legal context in which the investigation is always conducted also determines a number of characteristic properties. For example, in general the expert knows what hypothesis the prosecutor will present during the trial but usually not exactly what hypothesis the defender will bring forward. Moreover, the hypotheses can change at any time if new information becomes available or if the defence unexpectedly comes up with an alternative scenario. The expert can of course himself come up with alternative hypotheses. Once the hypotheses have been stated, the assessment of the prior odds is left to the judge. While most scientists want to make statements concerning the posterior odds of the hypotheses they are studying (for example: this is the probability that the patient has a certain illness, given the symptoms), according to the Bayesian theory described above, the forensic expert should restrict his statements to reporting the evidential value. The legal context also causes the forensic expert to be interested in a different type of error when reviewing hypotheses in the traditional manner. When comparing the refractive index of glass fragments found in the clothing of the suspect to the refractive index of samples of the smashed-in window at a burglary site, the usual null hypothesis is that the two populations do not differ (Rudin and Inman 2003). In general, the statistician is then interested in a type I error, the probability of rejecting the null hypothesis when it is actually true, and the procedure is aimed at controlling this error, for example keeping it smaller than 5% or 1%. This error is associated with the probability that the suspect is wrongfully acquited. Indeed, this is the probability that the expert concludes that the glass fragments do not come from the window while they in fact do. For experts this is not the most important error. It is more important to control the type II error, the conclusion that the glass fragments come from the window while this is not the case, which can lead to a wrongful conviction.

Other research in forensic statistics

Statistics is used much more often in legal applications than people realize. Sometimes it forms the basis for an expert's conclusion, for example in criminal psychology, which considers, among other things, the evidential value of recognition by witnesses, or of the result of a polygraph test (Van Koppen et al. 2002). The evidential value of scent identification tests with specially trained dogs is also in part based on statistical considerations (Schoon and Van Koppen 2002). A completely different area where statistics plays a role is in assessing the probability of recidivism in TBS patients (persons who have been admitted, involuntarily, to a forensic psychiatric hospital in the Netherlands) (De Ruiter 2002, Brand and Diks 2001). A great variety of questionnaires can be used for this, which combined with a clinical evaluation by the psychiatrist or psychologist lead to an estimation of this probability. Subsequently, this is used by the lawyer in his judgment of whether TBS must be extended. Probability theory also contributes to criminal law, for example in determining whether a game must be regarded as a game of skill or one of chance (Van der Genugten et al. 2001). Other examples are the probability analysis carried out in cases such as that of Bianca K., the child care worker who was accused of causing a special form of suffocation in a number of children whom she was responsible for at a daycare centre. A similar well-known case is that of Lucia de B., the nurse suspected of being responsible for a number of enigmatic deaths in her department. In this type of case, the size of the probability that certain observations are based only on chance plays an important role (Elffers 2003).

Solution: answer 2 is correct. The evidential value of the DNA match is increased by the database search. However, the probability that the matching person is the donor of the saliva trace may still be relatively small in such cases. Indeed, if the suspect is identified by a database search, there need not be any other evidence against the suspect. In other words, the prior odds may be extremely small, much smaller than in the situation where DNA is compared from a suspect who is identified by other evidence. Hence, the only thing special about a database search is that the other evidence may be completely missing, or only in favour of the suspect. See Meester and Sjerps (2003, 2004) and the references mentioned there.

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Society

Canonizing maths

What should one really know about math? In the Netherlands several lists appeared answering similar questions on different fields. The Dutch newspaper 'de Volkskrant' started such a list on science. If you had to compile such a list for mathematics, what would you argue to be indispensable?

Suppose you were asked to make a list of 50 topics (people/facts/phenomena) in mathematics and the natural sciences that every adult should know about. What would you choose? This question, always good for dynamic discussions at the coffee machine, became a reality on paper at the end of 2006 when a major Dutch newspaper de Volkskrant decided to publish such a list.

'Cause for action' was the creation of a similar list with topics from Dutch history, the so-called *Canon van Nederland*, instigated by the Dutch government to revive and standardize the teaching of history at primary schools. In addition to a lot of debate about the necessity of such a list, the Canon van Nederland became the source of a modest hype. Various web sites and magazines presented their own canons (lists of things they thought canonical) like the canon of Dutch 20th century movies, the canon of famous soccer players and, most elaborate and glamorous of all, the 'de Volkskrant Betacanon' of maths and natural sciences. The peculiar prefix 'beta' originates from an old Dutch school system, in which students had the choice to focus in their last two years of secondary education on languages ('alpha') or sciences ('beta'). The system had already been abandoned in the 1960s but not before leaving deep traces in the language.

A group of eight renowned professors was formed to compose the list of 50 topics that would make up the canon of maths and natural sciences, and every Saturday in 2007 one of these topics, ranging from the Big Bang to the modern toilet system and from tectonic movement to the concept of error, would be revealed. In an 800 word text, written by 'a young and enthusiastic researcher', the subject and its role in modern science were explained in the newspaper. A longer version of this text was presented on an accompanying website where, in the spirit of Wikipedia, the interested reader could make additions and corrections. Mathematics was represented by a modest number of five topics: algorithms, formulae, the normal distribution, chaos and, at the very start of the project, the number zero.

Of course one could wonder if after a year of reading informative and entertaining stories, the average Dutch newspaper reader does indeed know more about maths and science — and the answer would probably be no. However, the overwhelming number of enthusiastic responses, both on the website and in the daily lives of the authors, makes it clear that the lack of interest for science is not so widespread in Dutch society as is commonly believed.

The 50 de Volkskrant Bètacanon topics

Mathematics
The Number Zero, Algorithms, Symbols and Formulas, Errors in theories, Financial Mathematics, The Normal Distribution, Chaos Theory

Physics
Plate Tectonics, Energy, Quantum Physics, Climate and Weather, The Big Bang, The Standard Model, Newton, Einstein, Entropy, Oceanic currents, Electromagnetism, The Solar System, Precision in measurement

Biology and Chemistry
The Periodic Table, Avogadro, Darwin, The Brain, Cognition, Photosynthesis, Enzymes, The Ecosystem, The Pavlov reaction, Micro-organisms, Life expectancy, Catastrophies, DNA, Sex, Ancestors, Beri-beri, Language

Technology
Sanitary systems, The Transistor, The Nuclear Bomb, Plastic, GPS, Dikes, Agriculture, Robots, The Computer, The Telephone, Technology and City Development, The Bicyle, Nanotechnology

Problems

| Open Star Problems

Posing mathematical problems is a venerable tradition of the Koninklijk Wiskundig Genootschap. Now and again a star problem illuminates the problem section of the Nieuw Archief voor Wiskunde. The editors do not know any solutions to these problems. Below is a selection of star problems from the last twelve years for which no solutions have been submitted yet.
Whoever first sends in a correct solution before the July 1, 2009 deadline will receive a prize of 50 euros. Solutions can be sent to uwc@nieuwarchief.nl or to the address given below in the left-hand corner; submission by email (in LaTeX) is preferred.
Group contributions are welcome. Participants should repeat their name, address, affiliation and, if applicable, year of study at the beginning of each problem/solution. If you discover a problem has already been solved in the literature please let us know.
The problems and results can also be found on the problem section website www.nieuwarchief.nl/ps.

Problem 1$\star$ (proposed by J. van de Lune in NAW, vierde serie deel 14, no. 3, nov. 1996, pp. 429)
Let the continuous function $f_1 : (0,1] \to \mathbf{C}$ be such that

$$\int_0^1 f_1(t)\mathrm{d}t := \lim_{\epsilon\downarrow 0}\int_\epsilon^1 f_1(t)\mathrm{d}t$$

exists (and is finite) as an improper Riemann integral. Prove that f_1 has a unique extension to $f : \mathbf{R}^+ \to \mathbf{C}$ that is
1. continuous on $\mathbf{R}^+$,
2. differentiable on $(1,\infty)$ and satisfies the differential-difference equation

$$f'(x) = -\frac{1}{x}f(x-1) \qquad (x>1).$$

Also, determine

$$\lim_{x\to\infty} xf(x).$$

Finally, show that, if $\int_0^1 f_1(t)\mathrm{d}t = f_1(1)$, then the series $\sum_{n=1}^\infty nf(n)$ and the integral

$$\int_0^\infty f(t)\mathrm{d}t := \lim_{T\to\infty}\int_0^T f(t)\mathrm{d}t$$

both converge absolutely and have the same value.

Problem 2$\star$ (proposed by A. Szilárd-Károly in NAW, vierde serie deel 14, no. 3, nov. 1996, pp. 430)
Let $A_0A_1\dots A_n$ be an n–simplex and G_i be the centroid of the system formed by the given points except A_i. Denote by A_i' the intersection of the circumscribed hypersphere of the simplex and the line A_iG_i (for $i = 0,\dots,n$). Find the maximum value of the sum

$$\sum_{i=0}^n \frac{A_iG_i}{A_iA_i'}.$$

Problem 3$\star$ (proposed by G.W. Veltkamp in NAW, vierde serie deel 14, no. 3, nov. 1996, pp. 430)
Let A and B be $n\times n$ matrices over $\mathbf{C}$. Suppose that $\lim_{k\to\infty}(A^k + B^k)$ exists. Show that there exists $M \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$ such that $\lim_{k\to\infty}(A^k - kM)$ and $\lim_{k\to\infty}(B^k + kM)$ exist. Give necessary and sufficient conditions on A and B for M to be zero.

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Problems

| Open Star Problems

Problem 4⋆ (NAW, vijfde serie deel 2, no. 1, mar. 2001)
Let $p : [0,1] \to \mathbf{R}$ be a continuous function with $p(t) \geq 0$ for all $t \in [0,1]$ and $\int_0^1 p(t)\mathrm{d}t = 1$. Does the integral function $f : \mathbf{C} \to \mathbf{C}$ given by

$$f(z) := e^z - \int_0^1 p(t)e^{zt}\mathrm{d}t$$

have infinitely many zeroes?

Problem 5⋆ (NAW, vijfde serie deel 2, no. 3, sep. 2001)
Consider n $(n \times n)$ matrices with complex coefficients, A_i, such that for all i and j the i^{th} column of A_j is equal to the j^{th} column of A_i. Moreover each $\mathbf{C}$–linear combination of these n matrices is nilpotent (B is called nilpotent if $B^k = 0$ for some $k \leq 1$). Construct an arbitrary $n \times n^2$ matrix by placing the given n matrices one next to the other. Are the rows of this matrix dependent over $\mathbf{C}$?

Problem 6⋆ (NAW, vijfde serie deel 2, no. 3, sep. 2001)
Is there for every natural number N, a natural number k such that the ternary expension of k^2 contains no twos and at least N ones?

Problem 7⋆ (NAW, vijfde serie deel 3, no. 1, mar. 2002)
For $n = 1,2,3,\ldots$ we define the functions $\Phi_n : \mathbf{R} \to \mathbf{R}$ by $\Phi_n(x) = (2n)^x - (2n-1)^x + (2n-2)^x - (2n-3)^x + \ldots + 2^x - 1$. Prove or disprove that for all $x \in \mathbf{R}$ and for all $n = 1,2,3,\ldots$

1. $\Phi_n'(x) > 0$;
2. $\Phi_n''(x) > 0$.

What can be said about higher derivatives?

Problem 8⋆ (NAW, vijfde serie deel 3, no. 2, jul. 2002)
Can you cover an n–dimensional cube by n smaller cubes?

Problem 9⋆ (NAW, vijfde serie deel 3, no. 3, sep. 2002)
Does there exist a continuous surjection $f : [0,1] \to [0,1]^2$ such that every convex set has a convex image?
Reference: Pach and Rogers, *Partly convex Peano curves*, Bull. London Math. Soc. 15 (1983), no. 4, pp. 321–328.

Problem 10⋆ (NAW, vijfde serie deel 3, no. 3, sep. 2002)
Let $\mathbf{x}$ be a vector in $\mathbf{R}^{\mathbf{n}}$ with coordinates in $\{-1,1\}$ each randomly chosen with probability $\frac{1}{2}$. Let $\mathbf{y}$ be a fixed vector of unity in $\mathbf{R}^{\mathbf{n}}$. What is the probability that the inner product $\mathbf{x} \circ \mathbf{y}$ is smaller than 1?

Problem 11⋆ (NAW, vijfde serie deel 4, no. 1, mar. 2003)
Let V be the complex vector space of all functions $f : \mathbf{C} \to \mathbf{C}$. Let W be the smallest linear subspace of V with the properties:
1. the function $f(z) = z$ belongs to W,
2. for all $f \in W$, $|f| \in W$.
Does the function $f(z) = \overline{z}$ belong to W?
Remark: a conjecture in the theory of Boolean algebra can be reduced to this problem.

Problems

Open Star Problems

Problem 12⋆ (NAW, vijfde serie deel 4, no. 3, sep. 2003)
Determine the maximum area of a rectangle that can be covered by six disks of unit diameter.

Problem 13⋆ (NAW, vijfde serie deel 5, no. 1, mar. 2004)
For $x \in \mathbf{R}$ define

$$P_n(x) := n^n x \left((x+1)^{n+1} - 1 \right)^{n-1} - (n+1)^{n-1} \left((x+1)^n - 1 \right)^n$$

for $n \geq 2$. Is it true that this polynomial is of the form

$$P_n(x) = \sum_{k=n+2}^{n^2} c_{nk} x^k$$

with $c_{nk} > 0$ for $n+2 \leq k \leq n^2$?

Problem 14⋆ (NAW, vijfde serie deel 6, no. 1, mar. 2005)
Alice and Bob play a game. Alice places $n-1$ candles on a square cake. Bob places an extra candle in the bottom-left corner. Then, for each candle, he cuts a rectangular piece of cake such that the candle is at the bottom-left corner and no other candles are in the rectangle. Bob gets all these n pieces of cake.
1. Is it always possible for Bob to get more than half of the cake?
2. What is the optimal strategy for Alice to hold onto as much cake as possible?
Reference: Ponder This Challenge June 2004 http://domino.research.ibm.com/Comm /wwwr_ponder.nsf/Challenges/June2005.html.

Problem 15⋆ (NAW, vijfde serie deel 7, no. 3, sep. 2006)
Let $f(X) \in \mathbf{Q}[\mathbf{X}]$ be a polynomial of degree n with rational coefficients. Suppose that degree(gcd$(f(X), f^{(k)}(X))) > 0$ for $k = 1 \ldots n-1$. Prove or disprove $f(X) = a(X-b)^n$ for some rational numbers a, b. (See also arXiv:math.AC/0605090, 3 May 2006.)